



ANA PAULA MARTINS

# NOMINAL TALES OF (FOR) REAL ECONOMIES

## TASTE FOR INFLATION, TRADE MONEY, COLLATERALS AND RESERVE REQUIREMENTS

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# Preface

This research explores the real effects of the monetization of a stylized one-sector capital growth model driven by rational representative agents. On the one hand, it introduces the appropriate adjustment to the state variable dynamic accounting equation to reflect a (paper) cash-in-advance or trade money finance constraint in the presence of exogenous real reserve requirements. Such constraint embeds real convertibility - even if not for idle real reserves-, conformable with the role of money as both general means of payment, unit of account and store of value, and with the purchase of money at the inverse of the (real product) general price level. Commodity money then arises as the special case of a 100% (real…) required reserve ratio. On the other, it suggests generalizations of the state equation that (also) encompass product immobilization – production-before-expenditure - constraints, as well as delays, or even real losses, along the money creation process, that, as the CIA assumption, reflect on inventory stock rotation. Additionally, one concludes that efficient outcomes require the use of (at least) two policy instruments.

Two types of objective functions are considered: a standard accumulated discounted felicity function; and a point-wise utility function embedding bequest motives. Generalizations assuming taste for nominal growth at the utility level were staged for each case - taste for inflation may reflect psychological traits, compounding to and of similar nature to time discounting, implying distaste for increases in the (real) size of the nominal unit of account, working similarly to money illusion - possibly, to nominal discounting of utility over nominal arguments. Also, productivity enhancement due to nominal growth – working, say, through ease of inventory drainage – was simulated. Technically (mathematically), the hypotheses are convenient to generate non-negative growth of optimal per capita nominal money balances – and, therefore, prices – along the (and…) steady states. In real economies, price stability insures the constancy of money as measurement unit; such devices allowed to at least achieve a stable (non-negative) balanced inflation growth rate – but are consistent with residual barter trade of unsold merchandise. Nominal MIU was therefore also tested, as felicity functions combining real as nominal consumption as arguments.

The analysis relies on a discrete methodology. A storable good economy is briefly confronted with some of the structures. Contrast (and merger) with a high-powered money supply multiplier mechanism is also briefly outlined. Time interval between transactions, the money rotation period, is endogenised.

# Contents

# Introduction

Conventional production technology of standard growth models suggests no immobilization costs other than implied by capital depreciation and time unit definition. That is true for continuous or discrete specifications. In contrast, intertemporal monetary representations of the (macro)economic system invariably rely on some generalization of the Clower's constraint which, if taken on aggregate, either imposes some, or must preclude some conversion of real assets by nominal balances increase. It is the purpose of this research to model the consequences of the two - distinct - underlying processes.

The issue has been circumvented in a variety of ways, usually by assuming some return government net transfers – money conversion flows reverted to seigniorage. However, under those (and exogenous), money balances control would still hardly fall outside the individual's discretionary power: if more than possessed cash holdings even after the (nominal) transfer are desired, they can be requested from the banking system (through borrowing), undesired ones eliminated (through outstanding loan repayment without re-signing).

Another interpretation assumes that cash-in-advance is required prior to any transaction. Then each period, net increases in overall trade, calling for money emission, are immobilized. That means, say, if all existing currency was obtained through credit, new (net) loans would take one period to be granted – by the central bank -

and meanwhile production would be waiting. Or that all production is in fact "on hold" (yet, without depreciating: we have inventories of all future period expenditure but no – other than delay - storage costs) for one period till it gets exchanged for money – to then be transferred back to other goods and services. If one then translates the restriction into a conventional real capital state equation, one is confronted with the loss of approximately [1] the change in current period's output. A loss such as that can be rationalized by shopping time, or production-cum-trade waiting time requirements. Yet, standard real growth models do not seem to call for a production-before-expenditure (say, consumption and investment) assumption; on the other hand, we would rather think money would diminish the latter, not add them. But without some time frictions, people would not have a reason to hold money, cash: they could keep switching instantaneously real goods (interest-bearing deposits….) for money and vice-versa whenever needed: if the price of money is the inverse of the price level, the (private) opportunity cost of *holding* currency is the interest rate; money velocity should approach infinity; and this did not happen, not even with credit and other cards. That expenditure delays can occur due to borrowing constraints appears possible; and to that extent, that they interplay with the rotation of inventory stocks (and its velocity…).

The main novelty of this research is therefore to produce a sufficiently general constraint to generate, firstly, both a CIA-finance [2] one and a variable production and exchange waiting time. Secondly, simple mechanisms representing the efficiency in the (new) money creation process: from a simple almost neutral re-insertion to a high-powered money supply multiplier. And thirdly, require (additional) immobilization of real resources, akin to commodity money – i.e., encompass the existence *real* reserve requirements; a rise in these would work similarly to a tightening of monetary policy, or to a request for pledges in exchange for money loans. Such constraint, along with a closing transaction money demand relation, is then replaced in the simple Ramsey's (1928) growth model; an inventory state equation is superimposed - allowing an accountingly consistent interpretation of the finance constraint, both microeconomically as with the national income-expenditure identity. Money is introduced in the real model as a mirror, a representation, of existing assets – a

[1] Should the general price level be stable…

[2] The literature does not identify CIA modelling – which imposes monetization of consumption expenditure instead of the conventional transactions money demand equation - with the Clower's finance constraint; yet the latter really imposes some sort of advance condition…

measurement/conversion device - rather than an asset *per se* – people do not hold money for itself; rather, they hold real wealth through money: by holding money, they own a claim, a property title, over existing real assets in the economy, and when they pay for goods and services with money they in fact trade, exchange, (part of) wealth they own for them -, and relies (as CIA models do) on technical transaction requirements [3] – along with individual optimization – to justify its possession.

Obviously, the time costs implicit in the Clower's constraint tend to generate, in the presence of population growth, vanishing per capita nominal balances - with parallel continuous deflation; even a stably populated economy that experiences real per capita growth, say, induced by exogenous technical progress, cannot escape the latter in the long-run… This occurs for the efficient solution, endogenizing the general price level, as in the – inefficient – competitive equilibrium outcome - generating a Friedman-like rule. Mathematically, a switch to a Sidrauski's (1967) money-in-utility (here, felicity) function would not solve the problem – because it considers real, not nominal, balances as argument. In MIU models, money appears in the utility function due to the ease of transactions, to the flow of transaction services, it allows; yet, one could argue that only indirectly (after all, as for those of capital) would we experience such benefits: in a deterministic world, there would be no reason for liquidity to be overly desired *per se.*

One could contend that nominal [4] rather than real per capita money balances are valued along with real consumption – that will provide a sensible steady-state with no inflation; a more or less even weight of nominal and real consumption at the felicity function level also will. The assumption could capture individuals' awareness to level and changes of the "real" size of the nominal measurement unit. A less contemplated alternative is to consider the inflation rate itself as affecting either the accumulated utility function – and admit it to be able to generate similar (but of opposite sign) effects to the real rate of time preference –, or the periodic production function. In fact, the presence of inflation seems to generate uneven discounting in experimental episodes[5]; we might as well suggest individual maximization of an accumulated nominally discounted felicity function, the latter with

[3] Wang & Yip (1992) provide an interesting survey of pertinent literature.
[4] One can find previous modelling with nominal balances in utility in Benassy (1990).
[5] See Frederick, Loewenstein & O'Donoghue (2002) for recent references.

nominal quantities as arguments. Under a transactions money demand (technology), the weighting for output generates a nominal growth rate argument of the utility and/or the production function. Moreover, one could then find a – even if distant – (an additional…) justification for Taylor's type of monetary policy rules [6].

We retain a deterministic [7] and discrete context, and stage a representative agent economy – isolating the analysis from other public finance concerns (distribution, public goods, externalities…). The traditional methodology applied to the model, involving path and steady-state analysis, allows us to study the effects of changes of technological (or environmental) parameters and to address policy issues. Optimal allocations are no longer attainable under decentralized equilibria – the simple Clower constraint would involve intrinsic non-neutrality. These are nevertheless rich for economic interpretation: a q-theory of investment but also of labor contracts and money balances held by firms can now be inferred.

The exposition proceeds as follows: section 1 introduces the financial constraint in the basic representative agent dynamic problem. Mathematical implications of the introduction of the inventory state equation with the finance or modified finance constraint are explored in section 2. Short-run dynamics and steady-state properties of a general model with reserves and conversion delays are explored in section 3, including a storable good economy application. Exogenous technical progress suggesting balanced growth trajectories is analyzed in section 4. Preferences exhibiting "taste for nominal balance" and "taste for nominal growth" are introduced in section 5, nominal growth productivity effects, in section 6. Equilibrium pricing systems are devised in section 7. Section 8 contrasts some results with those applying to a BIU (KIU) model. A high-powered money supply multiplier effect is modelled in section 9. Some further qualifications - suggesting possible extensions - on public policy are forwarded in section 10. A final appraisal produces a concluding section.

[6] See Svensson (2003) for an appraisal. These usually stem from exogenous central bank's objectives.

[7] Precautionary motives are therefore not linked to, nor their study an immediate aim of, the research.

# 1. Nominal Conversion: The Banking System and the Financial Constraint

We will consider a representative – infinitely lived - agent economy, positioned at t = 0 and deciding for t = 1,2,…, where accumulated discounted felicity is maximized:

$$\sum_{t=1}^{\infty} \rho^t U(c_t) \qquad (1.1)$$

$c_t$ denotes (per capita, real) consumption in period t, ρ is the periodic discount factor [8]. Population grows - at an exogenous constant rate n [9], i.e., $L_t = (1 + n) L_{t-1}$ - $L_t$ denotes total population/labor force existing at time t (during period t…). If individuals value equally and additively the utility of all family members, present and future, ρ should be replaced by – or interpreted as - ρ = ρ' (1 + n) [10], where ρ' is the appropriate individual (unit) discount factor; that is, if (1.1) represents the objective function of an horizontal as vertical Benthamite

[8] I.e., ρ = 1 / (1 + ro) where ro is – has the status of a - the discount rate. Then, ro = (1 / ρ) - 1.

[9] For Portugal, 1953-1995 – using information from Pinheiro *et al.,* (1997) -, the average annual population growth rate was 0.45% (of employment, 0.77%) and virtually irrelevant. That may not be the case for sub-periods – and definitely not for other countries.

[10] See Barro & Sala-i-Martin (1995), p. 61 and ft. 4.

household and relevant decision unit, (proportional to) $\sum_{t=1}^{\infty} \rho'^{t} L_t$

$U(c_t) = L_0 \sum_{t=1}^{\infty} \rho'^{t} (1+n)^t U(c_t)$.

The representative consumer-producer must decide whether to produce investment goods, $i_t$, adding to his pre-existing (physical) capital stock, $k_{t-1}$ - the unit of which depreciates at rate d per period -, or consumption goods, $c_t$, during each period. The goods - made available at time t - are homogeneously generated by an aggregate CRS production function, $F(K_{t-1}, L_t)$, implying an average labor product one denoted by $y_t = f(k_{t-1})$, with $f(0) = 0$ and $f_k(k_{t-1}) > 0$ around the relevant range of $k_{t-1} = \frac{K_{t-1}}{L_t}$, the capital-labor ratio providing today's output. $k_{t-1} = \frac{K_{t-1}}{L_t}$ as in the Solow-Swan model [11]; all other per capita variables are defined in a consistent time basis of numerator and denominator.

The economy is a monetized one; there is a currency conversion requirement: "money buys goods, goods buy money, but goods don't buy goods" [12]- labor and capital services buy money, money buys goods, but the former cannot buy goods directly nor *vice-versa* - and cash must meet it. Then, each period – at the end of each period -, total product must be "monetized":

$$M_t = P_t f(k_{t-1}) \qquad (1.2)$$

$M_t$ denotes the aggregate nominal money stock at time t divided by $L_t$, labor/population, and $P_t$ the general price level – in nominal units - at which it is traded. Transactions are not made – not even implicitly… -continuously in time, and money is not instantaneous: if used at a transaction, it must be held – by either of the sides – for one period of time.

t is defined in the appropriate revolving payment period units [13] – where the unit of time coincides with the time elapse at which

[11] See Azariadis (1998), p. 4, for example.
[12] Clower (1967).
[13] In the models below, a transactions demand equation and payment rotation practices end up by ruling real cash-balances demand. The model would equally

additional – or less - money balances are required (and we assume the valid production function in the economy has the same time dimension) to trade *total* $y_t$. So, it would be as if transactions – corresponding nominal payments and/or accounting clearance - occur at discrete points in time, one time unit apart – and money velocity is one. Equation (1.2) establishes the demand for real cash-balances, which becomes completely determined by individuals discretion, once they determine (sequences of) $k_t$. Therefore, the equation is also one of general price level determination which, given nominal money balances – money supply -, adjusts real balances to output.

Due to (1.2), the inflation rate, $\pi_t = P_t / P_{t-1} - 1$, always approximates the growth rate of per capita money balances, $m_t = M_t / M_{t-1} - 1 = [(L_t M_t) / (M_{t-1} L_{t-1})] / (1 + n) - 1$, minus the growth rate of per capita product - or the growth rate of aggregate money balances minus the growth rate of production, i.e.:

$$(1 + \pi_t) = (1 + m_t) f(k_{t-2}) / f(k_{t-1}) \qquad (1.3)$$

To the extent that money is needed, changes in private holdings of cash-balances $dM_t$ - per capita money issuances at time t - can be purchased from the central authority – for a *physical counterpart*, that the consumer must therefore produce -, "priced" – as cash balances are traded in the economy - at the inverse of the general product price level, $P_t$. Then:

$$c_t + i_t + dM_t/P_t = f(k_{t-1}) \qquad (1.4)$$

and:

$$M_t = M_{t-1} / (1 + n) + dM_t \qquad (1.5)$$

apply with a money velocity equation $V M_{t'} = P_{t'} y_{t'}$ with t' generically defined and V as income transactions per unit of time – then one unit of time t should have length t'/V. Either formulation would be regarded as a technological one, dictated by the speed of business affairs – and the time unit to which the effective production function $f(k_{t-1})$ applies to. For Portugal, using data – from Pinheiro *et al.*, (1997) - covering 1953-1995, regression, without intercept, of annual per capita nominal GDP on (end of the year) per capita nominal aggregate yielded: for (at official, accounting price) gold reserves, 11.1021; for central bank total assets, 2.99862; for currency, 16.3533; for narrow money, M1, 4.06849; for M2, 1.42521.

Simultaneously, the central bank reinserts part of these $dM_t/P_t$'s – real goods - through *loans* - in cash, requested to the issuing authority -, $dB_t/Q_t$ [14] in the system, implicitly lending them at the prevailing rate - $Q_t$ is the price of investment in terms of private debt (bank credit) value; $dB_t$ denotes then credit issuances in period t; we will denote $dB_t/Q_t$ by $db_t$. Or it just bought assets through open market operations – if the assets were government debt titles, it financed government expenditures or transfers (it monetized - net - debt…). Then:

$$c_t + i_t + dM_t/P_t - dB_t/Q_t = f(k_{t-1}) \quad (1.6)$$

Leaving (other) public finance aside[15], we can think that $dM_t/P_t$ can be borrowed at the prevailing interest rate from the issuing authority and, with no delay, $dM_t/P_t = dB_t/Q_t$; if investment goods are instead purchased by the monetary authority for cash, then the central bank will receive capital income from its property; in any case, the authority is owned by all citizens, who ultimately collect the revenue and own the bank's assets. Money is a liability of the issuing authority; it can perform such operations to the extent that it is assumed to satisfy collateralized transactions: in the first case, by the investment goods on which account the loan was granted, in the second, money issuance is backed up by the real assets purchased by the bank... (If instead the government made transfers – granted subsidies - to provide the implicit re-insertion, in practice, it would have to issue public debt, backed by its future tax-raising ability, on the desired amount that would sell to the central bank to get the money; yet, that makes no difference to the real model.)

Capital evolves according to:

$$(1 + n)\, k_t = k_{t-1} + i_t - d\, k_{t-1} = (1 - d)\, k_{t-1} + f(k_{t-1}) - c_t + dB_t/Q_t - dM_t/P_t \quad (1.7)$$

At each trading-point in time - before trading - the economy's wealth-value is:

[14] This models how seigniorage is channelled back to private expenditure. For a closed representative agent economy one must assume it is somehow…

[15] Allowing monetary operations to be reversible without taxes...

$$M_t + S_{t-1} P_t / Q_t = P_t k_{t-1} (1- d) + P_t f(k_{t-1}) \qquad (1.8)$$

$S_{t-1} P_t / Q_t$ is the monetary value of assets not traded during period t. $S_{t-1} P_t / Q_t = P_t k_{t-1} (1- d)$.

Suppose there are frictions and only a fraction h of today's cash issuances meet immediate expenditure opportunities. Then:

$$db_t = dB_t/Q_t = h\ dM_t/P_t + (1 - h)\, dM_{t-1}/P_{t-1} (1 - d_h) / (1 + n) \qquad (1.9)$$

and only a fraction – h, an exogenous system parameter - of money balances creation get immediate application, with another parcel being delayed to next period. We allow for such parcel (investment good – say, c is perishable), waiting to be purchased, to depreciate at rate $d_h$, which may be higher or lower than d. We can assume that nominal transfers from (or direct purchases by…) the government would have immediate effect, being mirrored in the first term of (1.9), with remaining loans – requiring unavoidable credit assessment delays - being represented by the second term: a proportion h – then, a government policy instrument - of desired increase in money balances is provided by nominal government (net nominal) transfers.

As an alternative to (1.9), a longer effective cost can be imposed in the economy: of existing investment generated as cash-balance coverage, only a fraction h' is made operational in each period:

$$db_t = dB_t/Q_t = h'\, dM_t/P_t + h'\, dM_{t-1}/P_{t-1} (1 - h') (1 - d_h) / (1 + n) + h'\, dM_{t-2}/P_{t-2} (1 - h')^2 (1 - d_h)^2 / (1 + n)^2 + \ldots =$$

$$= h' [dM_t/P_t + (1 - h')\, dM_{t-1}/P_{t-1} (1 - d_h) / (1 + n) + (1 - h')^2 (1 - d_h)^2 dM_{t-2}/P_{t-2} / (1 + n)^2 + \ldots ] =$$

$$= [(1 - h') (1 - d_h) / (1 + n)]\, dB_{t-1}/Q_{t-1} + h'\, dM_t/P_t \qquad (1.10)$$

With part of money creation being made through nominal transfers able to meet immediate expenditure, we could have:

$$dB_t/Q_t = (1 - h') (1 - d_h) (dB_{t-1}/Q_{t-1} - Tr_{t-1}) / (1 + n) + h'\, dM_t/P_t + (1 - h') Tr_t \qquad (1.11)$$

In the aggregate, without frictions in the credit market or other, h, h' = 1, and $dM_t/P_t = dB_t/Q_t$. Money – fiat money, paper titles defined in nominal units – can be – or can hope to be rendered - completely neutral and $M_t$ indeterminate - and also $P_t$ but the indeterminacy is completely innocuous [16]. That is lost – and optimality of "competitive" price formation through (1.2) – whenever a portion of $dM_t/P_t$ and/or its lags affects individuals' real budget constraints. Yet, we keep the assumption that from period t-1 to period t, titles in amount per capita $M_{t-1}$ - currency – (must) circulate in the hands of the public: immediate – real… - re-insertion does not, in our research, mean instant money, that could be borrowed and returned immediately, at the same point in time; the distinction would not be important for the efficient allocation, but it obviously would for an equilibrium: theoretically, instant money involves no seigniorage.

Suppose no delays exist but money is commodity money – *transmutable* species - which cannot be converted back into other goods (nor yield utility) until *one* unit of time t has passed. Then, $dB_t/Q_t = 0$: $dM_t$ is transmuted into gold at time t. Of course "$dB_t/Q_t$" – $M_t$ - continues to exist – but in the hands of the public and being used for trade: at transactions time, apart from old capital, there exists $f(k_{t-1})$ and $M_t/P_t$ real worth of commodities in the market – the latter constituting a stock (more or less…) perpetually immobilized. Also, "gold" in the hands of the public may erode so that (1.5) is replaced by

$$M_t = (1 - d_r)\, M_{t-1} / (1 + n) + dM_t \qquad (1.12)$$

Prices are then defined in quantity of gold.

A central authority is then superfluous (moreover, money is its own direct "collateral"…)… People would deposit the gold in the treasury safe-boxes for convenience only. Convertibility but with floating nominal conversion rate of the "deposit slips" would be consistent with $dB_t/Q_t = 0$ (if the treasury kept the gold immobilized in full) in (1.7) – at trading time it would be as if everyone had to trade with gold.

[16] One could argue that money, as unit of account, should have a stable purchasing power – i.e., in the real world, such stability could be desirable. Then, a optimal endogenous policy should generate zero (or at least undetermined) inflation. Yet, a nominal currency with stably changing real value, with appropriate calculation by the public, would provide, in a deterministic world, exactly the same measurement service... The argument is pursued in sections 5 and 6 below.

Of course, economies are expected to function with convertible paper. Fiat money was invented to avoid the inherent immobilization cost, allowing the gold to be (physically…) lent – for "consumption" uses; (gold) reserve requirements could then be imposed for security reasons: to provide an insurance against over-issuance the public is allowed to claim the commodity (with a representative agent, he has no problem of credibility towards itself); or because money can only be issued at the end of the period, uncertainty or other (extraneous to the model) may require more cash-balances than titles effectively issued. They would have to be imposed to finite the reserve multiplier mechanism that would develop to gain control of money supply… Then banks - the central bank - keep a morsel rr [17] of issued papers as commodity in the treasury vault – but these titles were (initially) kept convertible. It is as if the reinsertion would become [18] - with no delays:

$$dB_t/Q_t = dM_t / P_t - rr\,[M_t / P_t - (M_{t-1}/ P_{t-1})\,(1 - d_r) / (1 + n)] \tag{1.13}$$

Or rather, $dB_t/Q_t = dM_t/P_t$ and part of it – amount $rr\,[M_t / P_t - (M_{t-1}/ P_{t-1})\,(1 - d_r) / (1 + n)]$ - is applied in reserve acquisition. Under a sluggish conversion mechanism such as (1.9), "net" reinsertion could be $h\,\{dM_t/P_t - rr\,[M_t / P_t - (M_{t-1}/ P_{t-1})\,(1 - d_r) / (1 + n)]\} + (1 - h)\,(1 - d_h)\,\{dM_{t-1}/P_{t-1} - rr\,[M_{t-1} / P_{t-1} - (M_{t-2}/ P_{t-2})\,(1 - d_r) / (1 + n)]\}$.

(If the nominal unit is, or is indexed to, a quantity of "gold" – in fact, real product in our simple one-sector economy… -, $M_{t-1}/ P_{t-1}$ should be replaced by $M_{t-1}/ P_t$, with reserve expenditure $rr\,[M_t / P_t - (M_{t-1}/ P_t)\,(1 - d_r) / (1 + n)]$ per period; the same applies to lagged mechanisms such as (1.9) if delays add to the process.)

Notice that such real "vault" reserves are idle pledge: remaining money can have collateral in the investment loans that the bank (or rather, the "gold" owners through the bank – that lent the gold to the bank, which then lent part of it to investors who transmuted it back to capital) – still… - concedes. And/or in (existing assets through) the tax ability the central government also detains.

[17] If $rr \neq 1$, a multiplier effect could develop in a world with deposits… We briefly discuss it at the end of section 9.2.

[18] $rr\,M_{t-1}\,(1 - d_h)$ should be added to the right hand-side of (7) to include these idle assets. Yet, $Q_t$ may be formed after (7) only.

In a deterministic – fully honest - economy, (immobile…) real reserve requirements have no theoretical rationale – rr would be zero; in practice, they would be important in maintaining central bank credibility and independence, even in closed economies. On the other extreme, it has been argued (Blanchard & Fisher, 1989) that a "competitive fiat money system would inevitably degenerate into a commodity money system": on the one hand, people would not accept to let go of seigniorage to a private entity; on the other, money would have to be fully convertible for paper-money issued by a private entity to be credible, and slight distrust, species price change eliciting private arbitrage, – that could outburst often in a changing economy… -would lead to conversion effectively occurring… In fact, a 100% coverage (rr > 1 would mean that some buffer "pocket gold" is also kept under the bed) closely approaches the commodity money setting (if also 100% required reserve ratio is imposed on commercial banks). Finally, for the period 1953-1995, the coefficient of the regression (without intercept) of Portuguese per capita central bank gold reserves (at official, accounting price – at market prices, the values would be even higher) on per capita currency circulation was 1.24734, and on high-powered money of 0.402327 [19] and they do not change much if we shorten the sample to the more recent periods – surely a compelling reason not to neglect its existence [20]. Of course, the value of gold may largely reflect its usefulness as a reserve – then, if there is a fixed amount of (not transmutable) gold in the economy, Z, the existence of gold reserve requirements has no impact on expenditure: its market price, call it $q_t$, would just adjust so that rr $L_t$ $M_t$ / $P_t$ = $q_t$ Z / $P_t$ is kept constant over time.

*Paper cash* reserve requirements – as those usually imposed on commercial banks' – have no role – nor a money supply multiplier - in this economy: recall that all money needs, $M_t$, are *currency*, chosen optimally – solicited upon request – by the public subject to a cash conversion constraint… – at most we can admit that they influence h… We will relax the assumption in section 9.

Also, a Keynesian government "money expenditure" could either be considered included in $dB_t/Q_t$; or added to (1.13) at the "expense" of real reserves (e.g., these are suddenly or partly allowed to be kept as securities, i.e., stockholdings of private

[19] On total central bank assets, of 0.231488; on M1, narrow money - currency plus current (checking) account deposits -, 0.293378; on M2, 0.102331.

[20] And possibly of why CIA modelling with similar effects to $dB_t$ / $Q_t$ = 0 may end up mimicking similar consequences…

capital – then part of the immobilization requirement would have disappeared…)

So far, we disregarded trading - or production delays – as those implicit in the Clower's or conventional cash-in-advance constraint – which takes the form of (1.4), without insertion. A rationale advanced for it is severe borrowing constraints faced by consumers (that to explain (1.4) we extend to investors): expenditure-makers are constrained to use the (nominal) income they received – hence, that existed - in the previous period and $P_t\, c_t + P_t\, i_t = M_{t-1} / (1 + n)$; factors are fully paid at time t, but only purchase with that revenue at the next trading point - or factors are only paid after the product is sold (and cannot borrow), even if at pre-agreed real rates. Nominal prices only go down when $dM_t > 0$ - and therefore average product increased – till $P_t$, formed after (1.2), equates the exchange rate that balances the value $dM_t$ and (because only $L_{t-1}\, M_{t-1}$ is purchased) the increase in product; as factors are fully paid in real terms at that price, producers are resilient to let prices fall any further – in any case, payment and expenditure are simultaneous, producers only realize that there is excess supply at the current price level when trading time closes, as time goes by.

Another, is that people do not foresee production changes before they actually occur in the market – at transaction clearing time, that occurs discretely; as they may not be satisfied with what has been produced, they only change balances after they confirm their willingness to buy – one would say that new products, innovations, are never found worthwhile borrowing to pay for them in the period they are launched... Yet another, is that to use money one has to request it one period of time in advance, or must hold the amount during one period; as it bears no interest, individuals/society must lose – pay in real goods -, are discouraged to produce, one unit of time of interest and money real depreciation for the utilized numeraire at a (final product) transaction.

Absence of delays is consistent with the production function interpretation of the standard real growth model [21] – in which time to build [22] or trade would suggest the use of

$$c_t + i_t + dy_t \;=\; f(k_{t-1}) \qquad (1.14)$$

[21] See Barro & Sala-i-Martin (1995) and Aghion & Howitt (1998) for recent surveys.

[22] Or rather, time to produce; the device is more general and simple than, say, Kydland & Prescott's (1982).

where $dy_t = y_t - y_{t-1} / (1 + n)$ rather than the commonly used identity, $c_t + i_t = f(k_{t-1})$; then, $c_t + i_t = f(k_{t-2})$ and current expenditure falls on previous period product - the hypothesis would have the same nature as that only past capital is operational for the current period production process, that $K_{t-1}$ rather than $K_t$ is used as argument of $F(., L_t) = L_t\, y_t$. The inclusion of $dy_t$ could as well meet a requirement of previous production relative to aggregate expenditure, consumption as investment – say, a production-in-advance type of hypothesis: if CIA assumes that "a seller/producer who sells his output this period for money will be able to use it only in the next period" [23], in a world with credit, we would more easily accept that what is produced this period is only available for consumption and investment next period - resources must be "allocated-in-advance" to the production process and immobilized until the final product is sold (that in (1.14) takes one period). Yet, one of the benefits of the use of money would be to offset, at least in part, such time costs. On the other hand, if they exist, their effects would hardly be monetized in a fiduciary world – that is, more appropriately, if (1.14) holds, we can just have $M_t = P_t\, f(k_{t-2})$, with factors also being paid with one period lag. The Clower's constraint would mirror, nevertheless, similar expected real effects. A 100% required real reserve ratio would approximate them as well in the models to follow – but only if reserves did not suffer real depreciation (or $dy_t$ also did).

An obvious generalization considered below includes loss $dy_t$ factored by g' (0 in usual models… We allow for depreciation); except in section 9, we assume it is included in rr – say rr = rr' + g', where rr' denotes the required real reserve ratio that the central bank follows.

Immobilizations – and ignore for the moment time-to-build – waiting to be sold imply the existence of stocks: $dM_t / P_t$ in a Clower finance constraint (1.4) represent, induce, change in inventories. Let inventory (stock) per capita be denoted by $z_t$ and depreciate at rate $d_h$; then a new state equation must be added to the general problem:

$$z_t = z_{t-1} (1 - d_h) / (1 + n) + dM_t / P_t \qquad (1.15)$$

[23] Eden (2005), p. 86.

And $z_t$ 0 – a real or effective “no-Ponzi-game” condition or constraint... Even if $d_h = 1$, we still should be aware of this constraint, lest we lose the meaning of an average product function, $f(k_{t-1})$; that is, if a positive $dM_t / P_t$ could represent production losses, hardly a negative value – adding to production out of the air…- could be justified.

Notice that total immobilization of resources from time t-1 to t implicit in (1.14) amounts to $f(k_{t-2})$ – the corresponding “slack” stock may not require modelling – constraining… they may just be assumed deducted from current production, $f(k_{t-1})$, and not necessarily adding to past inventories in (1.15) - because, naturally, $f(.) > 0$: if only these costs are involved, the identity (1.4) becomes $c_t + i_t = f(k_{t-2})$, which therefore exists. The same can be said for reserves, once, due to the price equation, their change is $rr\ [M_t / P_t - (M_{t-1}/ P_{t-1})\ (1 - d_r) / (1 + n)] = rr\ [f(k_t) - f(k_{t-2})\ (1 - d_r) / (1 + n)]$ – the stock is always a fixed proportion of current product per capita, $rr\ M_t / P_t = rr\ f(k_{t-1})$ – its depreciation having a similar nature as that of capital itself.

The conversion process – (1.9) or (1.10) imposes practical state variable restrictions, but rather implies public budget management through taxes, transfers and debt issuances; if positive changes in the money stock occur, all is well; but negative ones imply “economic overheating” and the goods the government would be supposed to exchange for the undesired money balances can only come at the expense of property in the hands of the public. That means $dM_t / P_t - db_t$ would be bounded: the government can but switch resources into the system to the extent such goods exist and are idle; even if it could use inventories some now and then, it would not be able to play that game systematically – that would mean resources would flourish all the time… Then, we can impose $dM_t / P_t - db_t \geq 0$ - government cannot “invent” goods in excess of $f(k_{t-1})$; or $dM_t / P_t \geq 0$ and then in the first re-insertion mechanism, if $dM_t / P_t - db_t < 0$, $dM_{t-1} / P_{t-1}$ must be (was) large… We can collapse both to $dM_t / P_t - db_t \geq - \mathrm{Max}\{(1 - h)\ dM_{t-1}\ (1 - d_h) / [(1 + n)\ P_{t-1}], 0\}$ or $dM_t / P_t \geq \mathrm{Min}\{- dM_{t-1}\ (1 - d_h) / [(1 + n)\ P_{t-1}], 0\}$; or admit that the mechanism works through inventories as well and imposes (independently, i.e., in absence of delays implicit in (15)…):

$$z_t = z_{t-1} (1 - d_h) / (1 + n) + dM_t / P_t - db_t \quad (1.16)$$

Again, $z_t \geq 0$...

Even if individuals can only purchase with previous period money balances and cannot - do not - borrow, at time t firms have to pay $M_t L_t$ of resources in money; whatever is not met by current purchases, must be requested from the central bank. This implies that a conversion mechanism can be coupled, accumulate, with the previous – Clower's - one. $dM_t/P_t = y_t - y_{t-1} / (1 + n) (P_t / P_{t-1})$ can therefore be factored by g – now including the nominal balances purchase; g = 2 admits the full effective immobilization of current period's output of the usual Clower constraint (with $dB_t/Q_t$ then deducted to capture the real re-insertion). We would require $z_t = z_{t-1} (1 - d_h) / (1 + n) + g\, dM_t / P_t - db_t \geq 0$; the national income identity has, therefore, a mirror representation involving change in inventories:

$$c_t + i_t + g\, dM_t/P_t - db_t = f(k_{t-1}) = c_t + i_t + z_t - z_{t-1} (1 - d_h) / (1 + n) \quad (1.17)$$

A value of g that differs from 2 when $db_t$ is included can reproduce different timing of payments and conversion operations; it might mimic a different time span than 1 of the effective production function, or capture the inverse of income velocity of money (currency) demand - a concept that the finance constraint replaces: rather, with it, the unit time period is the one for which the money velocity, exogenously and technically determined, is one – the time interval at which money would perform a rotation, complete a payment cycle of the goods that the economy produces, and changes in its size are/can be demanded (noticed). In appendix A, we use the finance constraint to suggest – a Baumol-Tobin type – endogenous determination of the appropriate time spanning period(s) between transaction points; in the text, we assume it given.

Lastly, notice that if money exchanges were performed continuously between t-1 and t – even if expenditure was only realized at the end of the (discrete) period -, one could argue that $dM_t/P_t$ should in fact be replaced by $\int_{t-1}^{t} \frac{d(\frac{M_u}{P_u})}{du} du$ or $d(\frac{M_t}{P_t})$

which equals $dy_t$ (That would be compatible with instant money…); then the money finance constraint gains real independence – and we can hope for money neutrality. Indivisibilities in the vertical production process implying accumulated flows being traded at discrete time intervals and/or at fixed price imply terms of the order of $dM_t$ / $P_t$ in the flow equation – that, because money does not pay interest, cannot be "sterilized" by individuals: a change in stock over a stock; that will imply radical changes in the optimal paths of the economy – and real consequences of nominal fluctuations as are attributed to staggered contracts and/or price rigidity.

We will stage three economies - one with commodity reserve requirements and MA(1) type of adjustment (9); another with the infinite – AR(1) type - adjustment (10). Finally, lag structure (9) is superimposed to the economy without capital but with a storable good. In later developments, we will concentrate on the first case. Except in section 7, we will focus on efficient allocations, on a central planner's view.

## 2. Short-Run Dynamics and Steady-State Properties: A Mathematical Note

Let us stage the simple finance constraint. The planner's problem is:

$$\underset{c_t, k_t, dM_t, M_t, P_t, z_t}{Max} \sum_{t=1}^{\infty} \rho^t \, U(c_t) \tag{2.1}$$

s.t: $(1 + n)\, k_t = (1 - d)\, k_{t-1} + f(k_{t-1}) - c_t - dM_t/P_t$ (2.2)

$M_t = M_{t-1} / (1 + n) + dM_t$ (2.3)

$M_t = P_t\, f(k_{t-1})$ (2.4)

$z_t = z_{t-1}\,(1 - d_h) / (1 + n) + dM_t / P_t$ (2.5)

$c_t \geq 0,\ k_t \geq 0,\ M_t \geq 0,\ z_t \geq 0$ (2.6)

Given $k_0$, $M_0$, $z_0$ (2.7)

As $dM_t/P_t = f(k_{t-1}) - f(k_{t-2})\, P_{t-1} / [P_t\,(1 + n)]$ there is always an advantage in providing for deflation – by (2.1), it would add to capital or consumption. In a steady-state, given (2.4) and that $z_t \geq 0$, an optimal solution would at best drive $dM_t / P_t$ to zero – reachable by setting $M_t / M_{t-1} = 1 / (1 + n)$. Out of the steady-state, monetary policy would be ascribed to "sterilize" inventories, after turning first period stocks into potential capital: one would exhaust or convert $z_t$, and set $M_t / M_{t-1} = 1 / (1 + n)$ – fix aggregate money balances - and the economy would be indistinguishable from that of the conventional real model – without inventories. In fact this is the solution we will get; let us prove that that is the case.

Let us replace the price determination equation. Then:

$$\underset{c_t, k_t, dM_t, M_t, z_t}{Max} \sum_{t=1}^{\infty} \rho^t \, U(c_t) \tag{2.7}$$

$$M_t = M_{t-1} / (1 + n) + dM_t \tag{2.8}$$

$$z_t = z_{t-1} (1 - d_h) / (1 + n) + f(k_{t-1}) \, dM_t / M_t \tag{2.9}$$

$$c_t \geq 0, k_t \geq 0, M_t \geq 0, z_t \geq 0; \text{ given } k_0, M_0, z_0 \tag{2.10}$$

The money stock must always be non-negative – in fact, one would want it to be strictly non-negative. This implies bounds for $dM_t$: $M_t = dM_t + M_{t-1} / (1 + n) > 0$ implies $dM_t > - M_{t-1} / (1 + n)$; $M_{t-1} / (1 + n) = M_t - dM_t > 0$ implies $dM_t < M_t$. Then, in the control problem, we can replace the restrictions on the sign on $M_t$ for those on $dM_t$: $- M_{t-1} / (1 + n) < dM_t < M_t$.

Likewise, $z_t \geq 0$ is going to require that $- z_{t-1} (1 - d_z) / (1 + n) \leq dM_t / P_t \leq z_t$. Yet, due to the previous paragraph and the price determination equation (21), $- M_{t-1} / [P_t (1 + n)] = - f(k_{t-1}) M_{t-1} / [M_t (1 + n)] \leq dM_t / P_t \leq M_t / P_t = f(k_{t-1})$. And at time t, the lower bound, i.e., $z_t \geq 0$, imposes $dM_t \geq - z_{t-1} (1 - d_h) M_t/[f(k_{t-1}) (1 + n)]$, requiring $M_t \geq M_{t-1} / [\{1 + z_{t-1} (1 - d_h) /[f(k_{t-1}) (1 + n)]\} (1 + n)]$.

So, in fact, change in inventories are bounded to be between $-\text{Min}\{z_{t-1} (1 - d_z), f(k_{t-1}) M_{t-1} / M_t\} / (1 + n) \leq dM_t / P_t \leq \text{Min}[z_t, f(k_{t-1})]$, or $- \text{Min}\{z_{t-1} (1 - d_z) M_t / f(k_{t-1}), M_{t-1}\} / (1 + n) \leq dM_t \leq M_t \text{ Min}[z_t / f(k_{t-1}), 1]$. That will also make the collapse of the two state equation of an optimal programme possible – yet, we delay that for later manipulations.

We can write the lagrangean form of the previous problem as:

$$\underset{c_t, k_t, dM_t, \lambda_t, M_t, \mu_t, z_t, \eta_t}{Max} L = \sum_{t=1}^{\infty} \rho^t \, U(c_t) + \sum_{t=1}^{\infty} \lambda_t \, [- (1 + n) \, k_t + (1 - d) \, k_{t-1} + f(k_{t-1}) - c_t -$$

$$- dM_t \, f(k_{t-1}) / M_t \,] + \sum_{t=1}^{\infty} \mu_t \,[- M_t + M_{t-1} / (1 + n) + dM_t] +$$

$$\sum_{t=1}^{\infty} \eta_t \,[- z_t + z_{t-1} (1 - d_h) / (1 + n) + f(k_{t-1}) \, dM_t / M_t \,] \qquad (2.10)$$

$$c_t \geq 0, k_t \geq 0, - \mathrm{Min}\{z_{t-1} (1 - d_h) M_t / f(k_{t-1}), M_{t-1}\} / (1 + n) \leq dM_t \leq M_t \, \mathrm{Min}[z_t / f(k_{t-1}), 1] \qquad (2.11)$$

In the current form, the resolution may then follow the maximum principles – we take that $c_t \geq 0$, $k_t \geq 0$ are satisfied and these will naturally follow interior solutions.

Transversality conditions must be imposed – replacing the requirement of a final level of the two variables [24], requiring $\lim_{t\to\infty} \lambda_t k_t = \lim_{t\to\infty} \lambda^t U_c(c_t) k_t = 0$, $\lim_{t\to\infty} \rho_t M_t = 0$ and $\lim_{t\to\infty} \mu_t z_t = 0$. We will assume transversality as SOC are always satisfied – the latter usually are for concave felicity and average product functions in real growth models.

Notice that the implied Hamiltonian would be linear in the control $dM_t$, usually generating bang-bang trajectories – switching over the limiting boundaries of the control – and/or singular solutions [25]... The resolution, nevertheless, would obey the rules of FOC of any static program applied to (2.10) – or other lagrangean versions. And that is the type of path we will conclude for…

We will suggest the interior solutions by replacing the restrictions:

$$\underset{k_t, M_t, z_t, \eta_t}{Max} \sum_{t=1}^{\infty} \rho^t \, U\{(1 - d) \, k_{t-1} + f(k_{t-1}) - (1 + n) \, k_t - [M_t - M_{t-1} / (1+n)] \, f(k_{t-1}) / M_t\} +$$

[24] For a finite horizon problem, t = 1,…, T, either the last (T-1-th) equations – FOC's - would be changed or we would require two terminal values - for T and T + 1 – of each of the two variables.

[25] See Clark (2005), for example. A well-known example occurs in the standard neoclassical one-sector growth model, our departing real economy, if felicity is linear – see Intriligator (1971), p. 413-415.

$$+ \sum_{t=1}^{\infty} \eta_t \{- z_t + z_{t-1} (1 - d_h) / (1 + n) + f(k_{t-1}) [M_t - M_{t-1} / (1 + n)] / M_t \} \qquad (2.12)$$

$k_t \geq 0$, $M_t \geq 0$, $z_t \geq 0$; given $k_{-1}$, $k_0$, $M_{-1}$, $M_0$, $z_0$

Restrictions were eliminated by successive replacement of “free” (non state) control variable. Let us ignore then the boundary constraints for the moment. F.O.C., along with the restriction, require, for t = 1, 2, 3,…:

$$\frac{\partial W}{\partial k_t} = \rho^t ( - (1 + n) U_c(c_t) + \rho U_c(c_{t+1}) [(1 - d)+ f'(k_t) \{1 - [M_{t+1} - M_t / (1 + n)] /M_{t+1}\}] ) + \eta_{t+1} f'(k_t) [M_{t+1} - M_t / (1 + n)] / M_{t+1} = 0 \qquad (2.13)$$

$$\frac{\partial W}{\partial M_t} = \{ \rho^t [ - U_c(c_t) (M_{t-1} / M_t^2) f(k_{t-1}) + \rho U_c(c_{t+1}) (1 / M_{t+1}) f(k_t)] + \eta_t f(k_{t-1}) (M_{t-1} / M_t^2) - \eta_{t+1} f(k_t) / M_{t+1} \} / (1 + n) = 0 \qquad (2.14)$$

$$\frac{\partial W}{\partial z_t} = - \eta_t + \eta_{t+1} (1 - d_h) / (1 + n) = 0 \qquad (2.15)$$

If an interior solution for $z_t$ were possible, (2.15) would imply $\eta_{t+1} / \eta_t = (1 + n) / (1 - d_h)$. On the one hand, (2.13) would suggest that then, for a steady-state, either

- $\eta_t$ is zero and the state equation irrelevant or redundant; (2.14) would imply that in an interior steady-state path $\rho^t / M_t$ would be constant – with $M_t$ changing at the negative rate $\rho - 1$ (or $\rho (1 + n) - 1$ if future generations are valued) - or vanish; as $\rho^t / M_t$ constant would require in (2.14) $\frac{\partial W}{\partial M_t} = [\rho^t / M_t] [\rho - 1/\rho ] = 0$, it becomes impossible: we would have to have a trivial solution for $M_t$, $M_t = 0$. Then $M_t / M_{t-1}$ is indeterminate; as the state equation must be complied with, $[M_t - M_{t-1} / (1 + n)] / M_{t-1} = 0$ requiring $M_t / M_{t-1} = 1 / (1 + n) - 1$.

- or in (2.13), $[M_t - M_{t-1} / (1 + n)] / M_{t-1} = 0$. As, for an interior steady-state path (2.14) requires that $\rho^t / M_t$ and $\eta_t / M_t$ grow at the same rate or vanish; as $\eta_t$ grows necessarily and $\rho^t$ decreases, that is impossible. That implies the trivial – corner - solution $M_t = 0$, requiring $\frac{\partial W}{\partial M_t} < 0$ in the steady-state.

Otherwise, $\frac{\partial W}{\partial z_t} < 0$ and we get immediately to $z_t = 0$ – or its bound. But then, the state equation implies $[M_t - M_{t-1} / (1 + n)] = 0$ and per capita money balances decrease at the population growth rate. Or, the terms multiplying $\eta_t$ and $\eta_{t+1}$ in (2.13) must zero and $[M_{t+1} - M_t / (1 + n)] = 0$ for the equality of the expression to zero to apply while $\rho^t / \eta_t$ does not tend to a constant…

If $z_0 < M_0 / P_0$, $z_t > 0$ will be more stringent than $M_t > 0$. $z_t = 0$ would be reach as quickly as possible and $z_t = z_{t-1} = 0$, as well as change in inventories, $dM_t / P_t = f(k_{t-1})\ dM_t / M_t = 0$. Then, because $M_t = M_{t-1} / (1 + n) + dM_t$ implies

$$M_t / P_t = f(k_{t-1}) = M_{t-1} / [P_t (1 + n)] + dM_t / P_t = M_{t-1}\ f(k_{t-1}) / [M_t (1 + n)] + dM_t / P_t \qquad (2.16)$$

if $dM_t / P_t = 0$ and change in inventories is annulled, $1 = M_{t-1} / [M_t (1 + n)]$: the aggregate money stock is stabilized and per capita money balances decrease at the rate of population growth: $M_t / M_{t-1} = 1 - 1/(1+n)$ in this case, we jump immediately to the Ramsey real world.

If $z_0 > M_0 / P_0$, because $z_t$ depreciates, we do not expect the opposite to occur and that $M_t$ could become stringent more rapidly than $z_t$. If it does, then $M_t$ stabilizes (along with prices in a steady-state) and $z_t$ decreases at rate $(1 - d_z) / (1 + n) - 1$, while $dM_t / P_t$ is “doomed” to be positive and a proportion $1 - 1 / (1 + n)$ of $f(k^*)$; or aggregate money balance stabilizes, $M_t$ decreases at the rate of population growth – or $1 / (1 + n) - 1$ - and $z_t$ decreases at rate $d_z$.

Notice that the fact that re-insertion is not allowed implies we cannot switch between inventories and the capital stock unless through money and the parallel dynamics of the two aggregates, money and inventories is therefore forceful…

Also, due to the fact that (after…) $z_t$ goes to zero, both state equations – of $z_t$ and $M_t$ - of the original form become redundant; given the previous reasoning, and as $z_t$ may be more stringent than $M_t$, the money state equation becomes redundant sooner than that of $z_t$. We might as well have worked with:

$$\underset{k_t, z_t}{Max} \sum_{t=1}^{\infty} \rho^t \ U\{(1 - d)\, k_{t-1} + f(k_{t-1}) - (1 + n)\, k_t - z_t + z_{t-1} (1 - d_h) / (1 + n) \} \qquad (2.17)$$

s.t.: $k_t \geq 0,\ z_t \geq 0$

while keeping in mind that $M_t = P_t\, f(k_{t-1})$, $M_t \geq 0$, and

$$M_t = M_{t-1} / (1+n) + P_t\, [z_t - z_{t-1} (1 - d_h) / (1 + n)] = M_{t-1} / (1 + n) + M_t\, [z_t - z_{t-1} (1 - d_h) / (1 + n)] / f(k_{t-1}). \qquad (2.18)$$

The linearity of the discrete Hamiltonian with (two) state equations - on $k_t$ and $z_t$ - underlying (2.17) in the implicit control $dz_t = z_t - z_{t-1} (1 - d_h) / (1 + n)$ is immediate: a bang-bang result is expected, with $z_t$ vanishing: using the bound in (2.11) – or setting $z_1 = 0$ on (2.9) -, $dM_1 = - z_0 (1 - d_h) M_1/[f(k_0) (1 + n)]$, implying $M_1 = M_0 / [\{1 + z_0 (1 - d_h) /[f(k_0) (1 + n)]\} (1 + n)]$; as then $z_1 = 0$, $M_t = M_{t-1} / (1 + n)$, t = 2,3,… Then, aggregate money balances are constant – the per capita stock of money decreasing at the population growth rate - after t = 2: we get the Samuelson (1958) rule.

Note that the two state equation become incompatible unless in trivial solutions. If in a particular solution $z_t > 0$, then the state equation on $z_t$ would become redundant – one of the two, any way. We therefore proceed to the opposite case: replace $dM_t / P_t$ on the problems and keep the previous conclusion in mind.

The old as the new form would point to a drive of change in real inventories to zero – implementable with $P_t$ and $M_t$ decreasing at rate 1/(1+n) - 1. The marginal product of capital follows, with consumption:

$$f'(k_t) = (1 + n)\, U_c(c_t) / [U_c(c_{t+1})\, \rho] - (1 - d) \qquad (2.19)$$

while

$$c_t = -(1 + n)\, k_t + (1 - d)\, k_{t-1} + f(k_{t-1})$$

that can be replaced in (2.19), generating the Ramsey real path – but for a starting capital stock $k'_0$ that includes pre-existing inventories: in the first state equation (i.e., for t = 1), $k_0$ (1 - d) is replaced by $k'_0$ (1 - d) where $k'_0$ solves $k'_0 (1 - d) = k_0 (1 - d) + z_0 (1 - d_h) / (1 + n) + f(k_0) - f(k'_0)$. It is as if initial inventories are transformed into capital immediately (and kept at zero afterwards). The economy reaches the steady-state for which:

$$f'(k^*) = (1 + n) / \rho - (1 - d) \qquad (2.21)$$

The dynamics would imply that simply by depressing prices, people are wealthier and therefore buying immediately becomes affordable - by curtailing money supply when the product rises and thus inducing a decrease in the price level, the problem of different timing of "usable" income receipt, constrained at time t to that obtained by $L_{t-1}$ individuals, but shared by $L_t$, and expenditure is solved; but then transfers or loan planning would avoid the problem (if used in the purchase of investment goods, capital, it would be worthwhile…). The mechanism could be useful, though, to reproduce the presence of exogenous borrowing constraints: which would be realistic for private consumption, usually not backed by credit – and as traditional CIA models assume; or, say, rules conditioning aggregate real borrowing to increase at previous period growth…

Even with transfers or free individual borrowing, some delays can still arise in the conversion process: we get to the proposed re-insertion mechanisms. Assume the simplest one with a lag and assume the absence of the borrowing constraint of the previous problem – and the state equation ruling inventories in now (1.16). Then the planner's problem is

$$\underset{c_t, k_t, dM_t, M_t, P_t, z_t}{Max} \sum_{t=1}^{\infty} \rho^t\, U(c_t)$$

s.t: $(1 + n)\, k_t = (1 - d)\, k_{t-1} + f(k_{t-1}) - c_t - (1 - h)\, dM_t / P_t + (1 - h)\, dM_{t-1}\, (1 - d_h) / [(1 + n)\, P_{t-1}]$ (2.22)

$M_t = M_{t-1} / (1 + n) + dM_t$ (2.23)

$M_t = P_t\, f(k_{t-1})$ (2.24)

$z_t = z_{t-1}\, (1 - d_h) / (1 + n) + (1 - h)\, dM_t / P_t - (1 - h)\, dM_{t-1}\, (1 - d_h) / [(1 + n)\, P_{t-1}]$ (2.25)

$c_t \geq 0,\ k_t \geq 0,\ M_t \geq 0,\ z_t \geq 0$; given $k_{-1}, k_0, M_{-1}, M_0, z_0$

The discrete Hamiltonian still exhibits a linearity in the control $dM_t$ - a bang-bang result could be expected – yet the lag structure is now much more complex. We can replace all the constraints in the objective function and derive:

$$\underset{k_t, M_t, z_t, \eta_t}{Max} \sum_{t=1}^{\infty} \rho^t\, U\{(1 - d)\, k_{t-1} + f(k_{t-1}) - (1 + n)\, k_t - (1 - h)\, [M_t - M_{t-1} / (1 + n)]\, f(k_{t-1}) / M_t + (1 - h)\, (1 - d_h)\, [M_{t-1} - M_{t-2} / (1 + n)]\, f(k_{t-2}) / [(1 + n)\, M_{t-1}]\} +$$
$$+ \sum_{t=1}^{\infty} \eta_t\, \{- z_t + z_{t-1}\, (1 - d_h) / (1 + n) + (1 - h)\, f(k_{t-1})\, [M_t - M_{t-1} / (1 + n)] / M_t - (1 - h)\, f(k_{t-2})\, [(1 - d_h) / (1 + n)]\, [M_{t-1} - M_{t-2} / (1 + n)] / M_{t-1}\} \quad (2.26)$$

FOC are

$$\frac{\partial W}{\partial k_t} = \rho^t\, ( - (1 + n)\, U_c(c_t) + \rho\, U_c(c_{t+1})\, [(1 - d) + f'(k_t)\, \{1 - (1 - h)\, [M_{t+1} - M_t / (1 + n)] / M_{t+1}\}] + \rho^2\, U_c(c_{t+2})\, f'(k_t)\, [(1 - d_h) / (1 + n)]\, (1 - h)\, [M_{t+1} - M_t / (1 + n)] / M_{t+1}\, ) + f'(k_t)\, \{\eta_{t+1}\, [M_{t+1} - M_t / (1 + n)] / M_{t+1} - \eta_{t+2}\, [(1 - d_h) / (1 + n)]\, [M_{t+2} - M_{t+1} / (1 + n)] / M_{t+2}\} = 0 \quad (2.27)$$

$$\frac{\partial W}{\partial M_t} = \rho^t\, \{ - (1 - h)\, U_c(c_t)\, (M_{t-1} / M_t^2)\, f(k_{t-1}) + \rho\, U_c(c_{t+1})\, \{(1 - h)\, (1 / M_{t+1})\, f(k_t) + (1 - h)\, (1 - d_h)\, (M_{t-1} / M_t^2)\, f(k_{t-1}) / (1$$

$+ n)\} - \rho^2 U_c(c_{t+2}) (1 - h) (1 - d_h) (1 / M_{t+1}) f(k_t) / (1 + n) + (1 - h) [\eta_t f(k_{t-1}) [(M_{t-1} / M_t^2) - \eta_{t+1} f(k_t) \{(1 / M_{t+1}) + [(1 - d_h) / (1 + n)] [(M_{t-1} / M_t^2)\} + \eta_{t+2} f(k_{t+1}) [(1 - d_h) / (1 + n)] (1 / M_{t+1}) ] \} / (1 + n) = 0$ (2.28)

$$\frac{\partial W}{\partial z_t} = - \eta_t + \eta_{t+1} (1 - d_h) / (1 + n) = 0 \qquad (2.29)$$

Again, for an interior solution for (2.28), $M_t$ had to grow faster than $(1 + n) / (1 - d_h)$ and $M_t$ is trivial. If $z_t = 0$ is kept, $(\frac{\partial W}{\partial z_t} < 0)$, capital evolves as in a real Ramsey world from t = 1,2,3,…., with starting capital stock $k'_0$ that solves $k'_0 (1 - d) + f(k'_0) = k_0 (1 - d) + z_0 (1 - d_h) / (1 + n) + f(k_0)$.

A zero inventory-driven monetary policy can be maintained following $z_1 = z_0 (1 - d_h) / (1 + n) + (1 - h) f(k_0) [M_1 - M_0 / (1 + n)] / M_1 - (1 - h) dM_0 (1 - d_h) / [(1 + n) P_0] = 0$ and – for $z_t = 0$ afterwards - $f(k_{t-1}) [M_t - M_{t-1} / (1 +n)] / M_t = f(k_{t-2}) (1 - d_h) [M_{t-1} - M_{t-2} / (1 + n)] / [M_{t-1} (1 + n)]$, a flow which will be rotating from t = 2,3,…

That implies a path for t = 2,3,…:

$$[M_t - M_{t-1} / (1 +n)] / M_t = [f(k_{t-2}) / f(k_{t-1})] [(1 - d_h) / (1 + n)] [M_{t-1} - M_{t-2} / (1 + n)] / M_{t-1} = [f(k_0) / f(k_{t-1})] [(1 - d_h) / (1 + n)]^{t-1} \{[M_1 - M_0 / (1 + n)] / M_1\} \qquad (2.30)$$

or

$$1 - 1 / [(1 + n) (1 + m_t)] = y_{t-1} / y_t \{1 - 1 / [(1 + n) (1 + m_{t-1})]\} [(1 - d_h) / (1 + n)]$$

with $\{[M_1 - M_0 / (1 + n)] / M_1\} f(k_0) = dM_0 (1 - d_h) / [(1 + n) P_0] - z_0 (1 - d_h) / [(1 + n) (1 - h)]$, provided the expression can be kept between $- M_{t-1} / (1 +n)] / M_t$ and 1 (if $z_0 (1 - d_h) / [(1 + n) (1 - h)] - (1 - h) dM_0 (1 - d_h) / [(1 + n) P_0] > 0$, $M_t$ will decrease from $t = 0$ to $t=1$, i.e., $[M_1 - M_0 / (1 + n)] < 0$, and, necessarily, the second bound will be complied with) and thus $M_t \geq 0$. We can manipulate the expression to:

$$M_t = [M_{t-1} / (1 +n)] / (1 - [f(k_0) / f(k_{t-1})] [(1 - d_h) / (1 + n)]^{t-1} \{[M_1 - M_0 / (1 + n)] / M_1\}) =$$

$$= [M_{t-1} / (1 +n)] / (1 - [f(k_0) / f(k_{t-1})] [(1 - d_h) / (1 + n)]^{t-1} \{[M_1 - M_0 / (1 + n)] / M_1\}) \qquad (2.31)$$

or

$$1 + m_t = [1 / (1 + n)] / (1 - [f(k_0) / f(k_{t-1})] [(1 - d_h) / (1 + n)]^{t-1} \{[M_1 - M_0 / (1 + n)] / M_1\}) - 1 \qquad (2.32)$$

Then for $M_t \geq 0$, $[f(k_0) / f(k_{t-1})] [(1 - d_h) / (1 + n)]^{t-1} \{[M_1 - M_0 / (1 + n)] / M_1\} \leq 1$ for $t = 2,3,\ldots$ For $t = 2$, it requires that $dM_0 (1 - d_h) / [(1 + n) P_0] - z_0 (1 - d_h) / [(1 + n) (1 - h)] \leq f(k_1) (1 + n) / (1 - d_h)$. Stability is guaranteed because, $-1 \leq [(1 - d_h) / (1 + n)] \leq 1$: in the steady-state, the rate of change of $M_t$ converges to $1 / (1 + n) - 1$. In the short-run, for given t, $m_t$ increases – similarly to a Phillips curve - with $f(k_{t-1})$ iff $M_1 - M_0 / (1 + n) < 0$; more precisely, it increases with $f(k_{t-1}) / [(1 - d_h) / (1 + n)]^{t-1}$.

It is easy to show – we will prove it for the more complex case of section 3.1 – that targeting $z_t = 0$ and $[M_t - M_{t-1} / (1 + n)] = 0$ for $t = 2,3,\ldots$ allowing $z_1 > 0$ is not better, in real terms, than the current policy.

By targeting null inventories, the central authority would reach the same objective as with hypothetical money transfers – we are assuming that at most they can only be used in part (h of issued currency)… Time losses are neutralized. But money balances – and prices - vanish, a quite unpractical outcome.

Adding taste for real balances as a second argument of the utility function – the usual MIU case - would not help us much, given that it amounts to introduce output itself as directly valued in felicity: the efficient long-run path would also imply a decreasing price level in the presence of population growth. Introducing nominal money could – we leave its discussion for section 5.

Imposing a real official reserve requirement will induce real but not significant nominal effects: it will not change the pattern of the optimal monetary policy. rr $\{f(k_{t-1}) - f(k_{t-2}) / [(1 - d_r) (1 + n)]\}$ is now deducted from the capital state equation – and will be accounted for in the real dynamic path.

In some subsections, we will append the restriction $- f(k_{t-2}) / (1 + n) \leq f(k_{t-1})\, dM_t/M_t \leq f(k_{t-1})$or equivalent to cover for the inventory NPG one; the left bound implies that existing inventories cannot exceed previous period production – we assume it would have perished… Yet, we will not forget the effective bound it tries to represent – and look for paths where the new - artificial - bound is not stringent.

# 3. Short-Run Dynamics and Steady-State Properties

## 3.1. Lagged Investment and Reserve Requirement

If we combine the three mechanisms of the previous section, the planner's problem becomes:

$$\underset{c_t, k_t, dM_t, M_t, db_t, P_t, z_t}{Max} \sum_{t=1}^{\infty} \rho^t U(c_t)$$

s.t: $(1+n)\,k_t = (1-d)\,k_{t-1} + f(k_{t-1}) - c_t + db_t - g\,dM_t/P_t - rr\,[M_t / P_t - (M_{t-1}/P_{t-1})(1-d_r)/(1+n)]$ (3.1)

$db_t = h\,dM_t/P_t + (1-h)\,dM_{t-1}/P_{t-1}\,(1-d_h)/(1+n)$ (3.2)

$M_t = M_{t-1}/(1+n) + dM_t$ (3.3)

$M_t = P_t\,f(k_{t-1})$ (3.4)

$z_t = z_{t-1}\,(1-d_h)/(1+n) + (g-h)\,dM_t/P_t - (1-h)\,dM_{t-1}\,(1-d_h)/[(1+n)\,P_{t-1}]$ (3.5)

$c_t \geq 0$, $k_t \geq 0$, $M_t \geq 0$, $z_t \geq 0$; given $k_{-1}$, $k_0$, $M_{-1}$, $M_0$, $z_0$

In lagrangean form with some replacements (db in (3.7) should be understood replaced by (3.2)):

$$\underset{c_t, k_t, dM_t, \lambda_t, M_t, \mu_t, z_t, \eta_t}{Max} \quad L = \sum_{t=1}^{\infty} \rho^t \, U(c_t) + \sum_{t=1}^{\infty} \lambda_t \{- (1 + n) k_t + (1 - d)$$

$k_{t-1} + f(k_{t-1}) - c_t -$

$- (g - h) \, dM_t \, f(k_{t-1}) / M_t + (1 - h) \, [(1 - d_h) / (1 + n)] \, dM_{t-1}$

$$f(k_{t-2}) / M_{t-1} - rr \, [M_t / P_t - (M_{t-1} / P_{t-1}) (1 - d_r) / (1 + n)]\} + \sum_{t=1}^{\infty}$$

$\mu_t \, [- M_t + M_{t-1} / (1 + n) + dM_t] +$

$$+ \sum_{t=1}^{\infty} \eta_t \, \{- z_t + z_{t-1} (1 - d_h) / (1 + n) + (g - h) \, f(k_{t-1}) \, dM_t /$$

$$M_t - (1 - h) \, f(k_{t-2}) \, [(1 - d_h) / (1 + n)] \, dM_{t-1} / M_{t-1} \} \qquad (3.6)$$

$$- \text{Min}\{z_{t-1} (1 - d_h) \, M_t / f(k_{t-1}) - db_t \, M_t / f(k_{t-1}), M_{t-1}\} / (1 + n) \leq$$

$$dM_t \leq M_t \, \text{Min}[z_t / f(k_{t-1}) + db_t \, M_t / f(k_{t-1}), 1] \qquad (3.7)$$

$c_t \geq 0, \ (k_t \geq 0,)$

In the current form, the resolution may follow the maximum principles. Notice that the implied Hamiltonian would be linear in the control(s) $dM_t$, as in section 2: even losing the inventory state equation, we will get to the same path as before, with $dM_t$ immediately set to zero (after z) and per capita money balances increasing at the rate of population growth.

We will suggest the interior solutions by replacing the restrictions:

$$\underset{k_t, M_t, z_t, \eta_t}{Max} \quad \sum_{t=1}^{\infty} \rho^t \;\; U\{(1 - d) \, k_{t-1} + f(k_{t-1}) - (1 + n) \, k_t - (g - h) \, [M_t -$$

$$M_{t-1} / (1 + n)] \, f(k_{t-1}) / M_t + (1 - h) \, [(1 - d_h) / (1 + n)] \, [M_{t-1} - M_{t-2} / (1 + n)] \, f(k_{t-2}) / M_{t-1} - rr \, [f(k_{t-1}) - f(k_{t-2}) (1 - d_r) / (1 + n)]\} +$$

$$+ \sum_{t=1}^{\infty} \eta_t \, \{- z_t + z_{t-1} (1 - d_h) / (1 + n) + (g - h) \, f(k_{t-1}) \, [M_t -$$

$$M_{t-1} / (1 + n)] / M_t - (1 - h) \, f(k_{t-2}) \, [(1 - d_h) / (1 + n)] \, [M_{t-1} - M_{t-2} / (1 + n)] / M_{t-1} \} \qquad (3.8)$$

$k_t \geq 0, \ M_t \geq 0, \ z_t \geq 0; \ \text{given } k_{-1}, k_0, M_{-1}, M_0, z_0$

Restrictions were eliminated by successive replacement of "free" (non state) control variable. Let us ignore then the boundary constraints for the moment. F.O.C., along with the restriction, require, for t = 1, 2, 3,…:

$\frac{\partial W}{\partial k_t}$ = $\rho^t$ ( - (1 + n) $U_c(c_t)$ + ρ $U_c(c_{t+1})$ [(1 – d)+ f'($k_t$) {1 – rr – (g – h) [$M_{t+1}$ – $M_t$ / (1 + n)] /$M_{t+1}$}] + $\rho^2$ $U_c(c_{t+2})$ f'($k_t$){[(1 – $d_h$) / (1 + n)] (1 – h) [$M_{t+1}$ – $M_t$ / (1 + n)] / $M_{t+1}$ + rr (1 – $d_r$) / (1 + n) } ) +

+ $\eta_{t+1}$ (g – h) f'($k_t$) [$M_{t+1}$ – $M_t$ / (1 + n)] / $M_{t+1}$ - $\eta_{t+2}$ (1 – h) f'($k_t$) [(1 – $d_h$) / (1 + n)] [$M_{t+1}$ – $M_t$ / (1 + n)] / $M_{t+1}$ = 0 (3.9)

$\frac{\partial W}{\partial M_t}$ = $\rho^t$ { - (g – h) $U_c(c_t)$ ($M_{t-1}$ / $M_t^2$) f($k_{t-1}$) + ρ $U_c(c_{t+1})$ {(g – h) (1 / $M_{t+1}$) f($k_t$) + (1 – h) (1 – $d_h$) ($M_{t-1}$ / $M_t^2$) f($k_{t-1}$) / (1 + n)} - $\rho^2$ $U_c(c_{t+2})$ (1 – h) (1 – $d_h$) (1 / $M_{t+1}$) f($k_t$) / (1 + n) + [$\eta_t$ f($k_{t-1}$) (g – h) [($M_{t-1}$ / $M_t^2$) - $\eta_{t+1}$ f($k_t$) {(g – h) (1 / $M_{t+1}$) + (1 – h) [(1 – $d_h$) / (1 + n)] [($M_{t-1}$ / $M_t^2$)} + $\eta_{t+2}$ f($k_{t+1}$) (1 – h) [(1 – $d_h$) / (1 + n)] (1 / $M_{t+1}$) ] } / (1 + n) = 0 (3.10)

$\frac{\partial W}{\partial z_t}$ = - $\eta_t$ + $\eta_{t+1}$ (1 – $d_h$) / (1 + n) = 0 (3.11)

The system possesses identical properties to the second problem of section 2. We expect inventories to be annulled – (3.11) being no longer active and with terms factored by the lagrange multiplier and the expenditure leakages in $\frac{\partial W}{\partial k_t}$ to disappear.

In real terms, it would be as if we would tend to the steady-state solution of:

$\underset{k_t}{Max} \sum_{t=1}^{\infty} \rho^t$ U{(1 – d) $k_{t-1}$ + f($k_{t-1}$) - (1 + n) $k_t$ - rr [f($k_{t-1}$) – f($k_{t-2}$) (1 – $d_r$) / (1 + n)]} (3.12)

$c_t \geq 0$, $k_t \geq 0$, Given $k_{-1}$, $k'_0$

From FOC, we derive that capital and per capita consumption follow:

$$f'(k_t) = [(1+n)\, U_c(c_t) - \rho\, U_c(c_{t+1})\,(1-d)] / \{ \rho\, U_c(c_{t+1})\,(1-rr) + \rho^2\, U_c(c_{t+2})\; rr\,(1-d_r)/(1+n) \} \qquad (3.13)$$

and

$$c_t = (1-d)\,k_{t-1} + f(k_{t-1}) - (1+n)\,k_t - rr\,[f(k_{t-1}) - f(k_{t-2})\,(1-d_r)/(1+n)] \qquad (3.14)$$

with dynamics similar to the well-known Ramsey path – slightly more contracted because reserve formation depress the economy in the same way as capital depreciation does -, with an initial stock $k'_0$ that solves $k'_0\,(1-d) + f(k'_0)\,(1-rr) = k_0\,(1-d) + z_0\,(1-d_h)/(1+n) + f(k_0)\,(1-rr)$.

The zero-inventory policy would require setting $M_1$ according to:

$$z_1 = z_0\,(1-d_h)/(1+n) + g\, dM_1/P_1 - db_1 = 0$$
$$= z_0\,(1-d_h)/(1+n) + (g-h)\,[M_1 - M_0/(1+n)]/[M_1\, f(k_0)] - (dM_0/P_0)\,(1-h)\,(1-d_h)/(1+n) \qquad (3.15)$$

Afterwards, $z_t = 0$ for $t = 2,3,\ldots$ implies then $(g-h)\,dM_t/P_t = (1-h)\,(dM_{t-1}/P_{t-1})\,(1-d_h)/(1+n)$ or:

$$(g-h)\,[M_t - M_{t-1}/(1+n)]\,f(k_{t-1})/M_t = [(1-h)\,(1-d_h)/(1+n)]\,[M_{t-1} - M_{t-2}/(1+n)]\,f(k_{t-2})/M_{t-1} \qquad (3.16)$$

or

$$M_t = [M_{t-1}/(1+n)] / (1 - [f(k_0)/f(k_{t-1})]\, \{[(1-h)/(g-h)]\,[(1-d_h)/(1+n)]\}^{t-1}\, \{[M_1 - M_0/(1+n)]/M_1\}) =$$

$M_t / M_{t-1} = 1 + m_t$, at given t, increases with $f(k_{t-1})$ iff $[M_1 - M_0/(1+n)] < 0$.

One can easily show that an alternative path that would target $z_2 = 0 = z_1\,(1-d_h)/(1+n) + (g-h)\,f(k_1)\,[M_2 - M_1/(1+n)]/M_2 - (1-h)\,(1-d_h)\,[M_1 - M_0/(1+n)]\,f(k_0)/[(1+n)\,M_1]$, allowing $z_1 = z_0\,(1-d_h)/(1+n) + (g-h)\,f(k_0)\,[M_1 - M_0/(1+$

n)] / $M_1$ - (1 – h) $dM_0$ (1 – $d_h$) / [(1 + n) $P_0$] and fixing the aggregate money stock after t = 1, i.e., let [$M_2$ – $M_1$ / (1 + n)] = 0 as afterwards would be inferior. With it, at t=1, the capital state equation of the Ramsey's world (but with reserves…) would be added of $z_0$ (1 – $d_h$) / (1 + n) – $z_1$, and at t = 2, of $z_1$ (1 – $d_h$) / (1 + n); as the latter has to be non-negative (requiring (1 – h) $dM_0$ / $P_0$ > $z_0$ > 0), as it depreciates, we would rather have inserted – not deducted - it in the first period – i.e., we be better-off just inserting $z_0$ (1 – $d_h$) / (1 + n) at t = 1 and let $z_1$ = 0 = $z_t$ afterwards.

$k_t$ is that of the path implied by (3.13) and (3.14). This means that monetary policy is "at the real service" – with the real sector variables' path being determined independently and then imputed to generate the (zero-inventory) money supply trajectory.

In a steady state, (3.16) implies that [$M_t$ – $M_{t-1}$ / (1 + n)] would tend to zero – and $M_t$ changes at rate m* = 1 / (1 + n) – 1 - the Samuelson (1958) rule, with a constant aggregate money stock. The real per capita consumption and capital would obey:

$$f'(k^*) = [(1+n) - \rho(1-d)] / \{\rho(1-rr) + \rho^2 [rr(1-d_r)/(1+n)]\} \quad (3.17)$$

and

$$c^* = f(k^*)\{1 - rr[1-(1-d_r)/(1+n)]\} - (n+d)k^* \quad (3.18)$$

Then, k* increases (the right hand-side of (3.17) decreases) with ρ. It decreases with n, d, rr, and $d_r$.

Note that vanishing money balances in the steady-state stem from the existence of terms associated with ($M_t$ – $M_{t-1}$) / $P_t$ – or its lags - of the felicity function and not from the official reserve requirement. Instead, the latter produces real effects on the steady-state level of capital.

### 3.2. Infinite Lag Adjustment of Investment Loans

Let us stage distributed lag form (1.10) for re-insertion. As $dB_t/Q_t$ = (1 - h') $dB_{t-1}/Q_{t-1}$ (1 - $d_h$) / (1 + n) + h' $dM_t/P_t$, the capital stock state equation becomes:

$$(1+n)\ k_t = (1-d)\ k_{t-1} + f(k_{t-1}) - c_t + dB_t/Q_t - g\ dM_t/P_t - rr\ [f(k_{t-1}) - f(k_{t-2})(1-d_r)/(1+n)] =$$

$= (1 - d)\, k_{t-1} + f(k_{t-1}) - c_t + (1 - h')\, dB_{t-1}/Q_{t-1}\, (1 - d_h) / (1 + n) - (g - h')\, dM_t / f(k_{t-1}) / M_t - rr\, [f(k_{t-1}) - f(k_{t-2})\, (1 - d_r) / (1 + n)]$ (3.19)

Four state equations are now necessary:

$$\underset{c_t, k_t, dM_t, M_t, db_t, z_t}{Max} \sum_{t=1}^{\infty} \rho^t U(c_t)$$

$(1 + n)\, k_t = (1 - d)\, k_{t-1} + f(k_{t-1}) - c_t + db_t - g\, dM_t\, f(k_{t-1}) / M_t - rr\, [f(k_{t-1}) - f(k_{t-2})\, (1 - d_r) / (1 + n)]$ (3.20)

$M_t = M_{t-1} / (1 + n) + dM_t$ (3.21)

$db_t = (1 - h')\, db_{t-1}\, (1 - d_h) / (1 + n) + h'\, dM_t\, f(k_{t-1}) / M_t$ (3.22)

$z_t = z_{t-1}\, (1 - d_h) / (1 + n) + g\, dM_t / P_t - db_t$ (3.23)

$c_t \geq 0,\ k_t \geq 0,\ M_t \geq 0,\ z_t \geq 0;$ given $k_{-1}, k_0, M_0, dM_0, db_0, z_0$

In lagrangean form:

$$\underset{c_t, k_t, dM_t, M_t, \lambda_t, db_t, \mu_t, z_t, \eta_t}{Max} L = \sum_{t=1}^{\infty} \rho^t U(c_t) + \sum_{t=1}^{\infty} \lambda_t \{(1 + n)\, k_t - (1 - d)\, k_{t-1} - f(k_{t-1}) + c_t - db_t + g\, dM_t\, f(k_{t-1}) / M_t + rr\, [f(k_{t-1}) - f(k_{t-2})\, (1 - d_r) / (1 + n)]\} + \sum_{t=1}^{\infty} \mu_t\, [- M_t + M_{t-1} / (1 + n) + dM_t] +$$

$$+ \sum_{t=1}^{\infty} \nu_t \{db_t - [db_{t-1}\, (1 - h')\, (1 - d_h) / (1 + n) + h'\, dM_t\, f(k_{t-1}) / M_t]\} + \sum_{t=1}^{\infty} \eta_t \{- z_t + z_{t-1}\, (1 - d_h) / (1 + n) + g\, f(k_{t-1})\, dM_t / M_t - db_t\}$$ (3.24)

At most, we can simplify the problem to:

$$\underset{c_t, k_t, dM_t, M_t, db_t}{Max} \sum_{t=1}^{\infty} \rho^t\, U\{(1 - d)\, k_{t-1} + f(k_{t-1}) - (1 + n)\, k_t + db_t - g\, [M_t - M_{t-1} / (1 + n)]\, f(k_{t-1}) / M_t - rr\, [f(k_{t-1}) - f(k_{t-2})\, (1 - d_r) / (1 + n)]\}$$ (3.25)

s.t.: $db_t = (1 - h')\, db_{t-1}\, (1 - d_h) / (1 + n) + h'\, [M_t - M_{t-1} / (1 + n)]\, f(k_{t-1}) / M_t$ (3.26)

$$z_t = z_{t-1} (1 - d_h) / (1 + n) + g\ f(k_{t-1}) [M_t - M_{t-1} / (1 + n)] / M_t - db_t \tag{3.27}$$

$c_t \geq 0$, $M_t \geq 0$, $z_t \geq 0$; given $k_{-1}$, $k_0$, $M_0$, $dM_0$, $db_0$, $z_0$

With lagrangean form:

$$\underset{k_t, M_t, db_t, \nu_t}{Max} \sum_{t=1}^{\infty} \rho^t\ U\{(1 - d)\ k_{t-1} + f(k_{t-1}) - (1 + n)\ k_t + db_t - g\ [M_t - M_{t-1} / (1 + n)]\ f(k_{t-1}) / M_t] - rr\ [f(k_{t-1}) - f(k_{t-2})\ (1 - d_r) / (1 + n)]\} +$$

$$+ \sum_{t=1}^{\infty} ?_t\{db_t - db_{t-1}\ (1 - h')\ (1 - d_h) / (1 + n) - h'\ [M_t - M_{t-1} / (1 + n)]\ f(k_{t-1}) / M_t\} +$$

$$+ \sum_{t=1}^{\infty} ?_t\ \{- z_t + z_{t-1}\ (1 - d_h) / (1 + n) + g\ f(k_{t-1})\ [M_t - M_{t-1} / (1 + n)] / M_t - db_t\} \tag{3.28}$$

$c_t \geq 0$, $M_t \geq 0$, $z_t \geq 0$; given $k_{-1}$, $k_0$, $M_0$, $db_0$, $z_0$

First-order conditions include the restrictions, (3.26) and (3.27) and:

$$\frac{\partial W}{\partial k_t} = \rho^t\ \{ - (1 + n)\ U_c(c_t) + \rho\ U_c(c_{t+1})\ [(1 - d) + f'(k_t)\ \{1 - rr - g\ [M_{t+1} - M_t / (1 + n)] / M_{t+1}\}] + \rho^2\ U_c(c_{t+2})\ f'(k_t)\ rr\ (1 - d_r) / (1 + n)\ \} - \nu_{t+1}\ f'(k_t)\ h'\ [M_{t+1} - M_t / (1 + n)] / M_{t+1} + \eta_{t+1}\ g\ f'(k_t)\ [M_{t+1} - M_t / (1 + n)] / M_{t+1} = 0 \tag{3.29}$$

$$\frac{\partial W}{\partial M_t} = \{\ \rho^t\ g\ [- U_c(c_t)\ (M_{t-1} / M_t^2)\ f(k_{t-1}) + \rho\ U_c(c_{t+1})\ (1 / M_{t+1})\ f(k_t)] + h'\ [-\nu_t\ (M_{t-1} / M_t^2)\ f(k_{t-1}) + \nu_{t+1}\ (1 / M_{t+1})\ f(k_t)] + \eta_t\ f(k_{t-1})\ g\ (M_{t-1} / M_t^2) - \eta_{t+1}\ f(k_t)\ g\ (M_t / M_{t+1})\ \} / (1 + n) = 0 \tag{3.30}$$

$$\frac{\partial W}{\partial db_t} = \rho^t U_c(c_t) + \nu_t - \nu_{t+1} (1 - h') (1 - d_h) / (1 + n) - \eta_t = 0 \quad (3.31)$$

$$\frac{\partial W}{\partial z_t} = - \eta_t + \eta_{t+1} (1 - d_h) / (1 + n) = 0 \quad (3.32)$$

The optimal monetary policy will exhaust inventories, implying that (3.32) will not hold in equality. Then:

$$z_t = z_{t-1} (1 - d_h) / (1 + n) + g\, dM_t / P_t - db_t = 0 \quad (3.33)$$

Than the real path of the economy is that of the previous sub-section; $c_t$ and $k_t$ is that of the path (3.13) and (3.14). It will imply setting $M_1$ according to:

$$\begin{aligned} z_1 &= z_0 (1 - d_h) / (1 + n) + g\, dM_1 / P_1 - db_1 = 0 \\ &= z_0 (1 - d_h) / (1 + n) + (g - h') [M_1 - M_0 / (1 + n)] / [M_1 f(k_0)] - db_0 (1 - h') (1 - d_h) / (1 + n) \end{aligned} \quad (3.34)$$

$z_t = 0$ for $t = 2,3,\ldots$ implies then $g\, dM_t / P_t = db_t$ or, from the definition of $db_t$:

$$(g - h') [M_t - M_{t-1} / (1 + n)] f(k_{t-1}) / M_t = db_{t-1} (1 - h') (1 - d_h) / (1 + n) \quad (3.35)$$

and

$$db_t = db_{t-1} (1 - h') (1 - d_h) / (1 + n) + h' [M_t - M_{t-1} / (1 + n)] f(k_{t-1}) / M_t \quad (3.36)$$

And the steady state for which the monetary aggregate converge requires - $db_t = db_{t-1}$ - $M_t$ changing at rate $m^* = 1 / (1 + n) - 1$.

## 3.3. A Storable Good Economy

A contrast can be made with an economy for which c is storable but there is no physical capital. Assume storable (yet, non-durable – i.e., totally exhausted after consumption -, but depreciable) manna falls from the sky at per capita rate $f_t$ per period (without technical progress – e.g., reasons for real productivity growth -, we will consider $f_t = f$, all t). Without a monetary constraint

$$\underset{c_t, k_{zt}}{Max} \sum_{t=1}^{\infty} \rho^t U(c_t)$$

$$k_{zt} = k_{zt-1} + f_t - c_t - d_z k_{zt-1} = (1 - d_z) k_{zt-1} + f_t - c_t \qquad (3.37)$$

$c_t \geq 0$, $k_{zt} \geq 0$; given $k_{z0}$

or

$$\underset{k_{zt}}{Max} \sum_{t=1}^{\infty} \rho^t \; U[(1 - d_z) k_{zt-1} + f_t - k_{zt}]$$

$k_{zt} \geq 0$; given $k_{z0}$

With a financial counterpart, and CIA and reserve requirements:

$$\underset{c_t, k_{zt}, dM_t, M_t, P_t, db_t, z_t}{Max} \sum_{t=1}^{\infty} \rho^t U(c_t)$$

$$\text{s.t: } (1 + n) k_{zt} = (1 - d_z) k_{zt-1} + f_t - c_t + db_t - g\, dM_t/P_t - rr\, [M_t / P_t - (M_{t-1}/ P_{t-1}) (1 - d_r) / (1 + n)] \qquad (3.38)$$

$$db_t = [h\; dM_t/P_t + (1 - h)\, dM_{t-1}/P_{t-1} (1 - d_h) / (1 + n)] \qquad (3.39)$$

$$M_t = M_{t-1} / (1 + n) + dM_t \qquad (3.40)$$

$$M_t = P_t f_t \qquad (3.41)$$

$$z_t = z_{t-1} (1 - d_h) / (1 + n) + (g - h)\, dM_t/ P_t - (1 - h) [(1 - d_h) / (1 + n)]\, dM_{t-1}/ P_{t-1} \qquad (3.42)$$

$c_t \geq 0$, $k_{zt} \geq 0$, $M_t \geq 0$, $z_t \geq 0$; given $k_{z-1}$, $k_{z0}$, $M_0$, $dM_0$, $P_0$, $db_0$, $z_0$

Again, if rr = 0 and g = h = 1 money is completely neutral.
The lagrangean after replacement of constraints becomes:

$$\underset{k_{zt}, M_t, z_t, \eta_t}{Max} \sum_{t=1}^{\infty} \rho^t \; U\{(1 - d_z) k_{zt-1} + f_t - (1 + n) k_{zt} - (g - h) [M_t - M_{t-1} / (1 + n)]\, f_t /M_t + \{(1 - h) [(1 - d_h)/(1 + n)] \} [(M_{t-1} - M_{t-2} / (1 + n)]\, f_{t-1} /M_{t-1} - rr\, [f_t - f_{t-1} (1 - d_r) / (1 + n)]\}$$

$$+ \sum_{t=1}^{\infty} \eta_t \{- z_t + z_{t-1} (1 - d_h) / (1 + n) + (g - h) f(k_{t-1}) [M_t - M_{t-1} / (1 + n)] / M_t - (1 - h) f(k_{t-2}) [(1 - d_h) / (1 + n)] [M_{t-1} - M_{t-2} / (1 + n)] / M_{t-1} \} \quad (3.43)$$

$c_t \geq 0$, $k_{zt} \geq 0$, $M_t \geq 0$, $z_t \geq 0$; given $k_{z0}$, $M_0$, $z_0$

$$\frac{\partial W}{\partial k_{zt}} = \rho^t [ - U_c(c_t) + ? U_c(c_{t+1}) (1 - d_z) ] = 0 \quad (3.44)$$

$$\frac{\partial W}{\partial M_t} = \rho^t \{- (g - h) U_c(c_t) (M_{t-1} / M_t^2) f_t + \rho U_c(c_{t+1}) [(g - h) (1 / M_{t+1}) f_{t+1} + (1 - h) [(1 - d_h)/(1 + n)] (M_{t-1} / M_t^2) f_t] - \rho^2 U_c(c_{t+2}) (1 - h) [(1 - d_h) / (1 + n)] (1 / M_{t+1}) f_{t+1} + [\rho_t f(k_{t-1}) (g - h) [(M_{t-1} / M_t^2) - \rho_{t+1} f(k_t) \{(g - h) (1 / M_{t+1}) + (1 - h) [(1 - d_h) / (1 + n)] [(M_{t-1} / M_t^2)\} + \rho_{t+2} f(k_{t+1}) (1 - h) [(1 - d_h) / (1 + n)] (1 / M_{t+1}) ] \} / (1 + n) = 0 \quad (3.45)$$

$$\frac{\partial W}{\partial z_t} = - \eta_t + \eta_{t+1} (1 - d_h) / (1 + n) = 0 \quad (3.46)$$

On one hand, $k_{z0}$ is added of $z_0 (1 - d_h) / (1 + n)$ in the optimal path derived for a case without leakages, provided $d_h > d_r$ – i.e., the storable good depreciates less rapidly than the monetary generated inventories. Then, the last condition becomes non-binding – storage was integrated - and the second redundant. $M_1$ is set according to:

$$z_1 = z_0 (1 - d_h) / (1 + n) + g\, dM_1 / P_1 - db_1 = 0$$
$$= z_0 (1 - d_h) / (1 + n) + (g - h) [M_1 - M_0 / (1 + n)] / (M_1 f_1) - (dM_0 / P_0) (1 - h) (1 - d_h) / (1 + n) \quad (3.47)$$

$z_t = 0$ for $t = 2,3,\ldots$ again implies $(g - h)\, dM_t / P_t = (1 - h) (dM_{t-1} / P_{t-1}) (1 - d_h) / (1 + n)$ or:

$$(g - h) [M_t - M_{t-1} / (1 + n)] f_t / M_t = [M_{t-1} - M_{t-2} / (1 + n)] f_{t-1} / M_{t-1} (1 - h) (1 - d_h) / (1 + n) \quad (3.48)$$

And the steady state for which the monetary aggregate converge - if $f_t = f_{t-1}$ - requires $M_t$ changing at rate $m^* = 1 / (1 + n) - 1$.

Also, (3.44) implies that whenever $k_{zt} > 0$:

$$U_c\{(1 - d_z) k_{zt-1} + f_t - (1 + n) k_{zt} - rr [f_t - f_{t-1} (1 - d_r) / (1 + n)]\} = \rho U_c\{(1 - d_z) k_{zt} + f_{t+1} - (1 + n) k_{zt+1} - rr [f_{t+1} - f_t (1 - d_r) / (1 + n)]\} (1 - d_z) \quad (3.49)$$

When $k_{zt} = 0$, $\rho U_c(c_{t+1}) (1 - d_z) < U_c(c_t)$ and:

$$c_t = (1 - d_z) k_{zt-1} + f_t - rr [f_t - f_{t-1} (1 - d_r) / (1 + n)] \quad (3.50)$$

If $f_t = f$, in the steady state of the real economy, we would expect $k_{zt} = 0$ and $c = f \{1 - rr [1 - (1 - d_r) / (1 + n)]\}$, and the first condition, (3.44) to be non-binding.

# 4. Technical Progress and Balanced Growth Paths.

Technical progress, may generate explosive paths. We shall analyze under which circumstances it may generate stable growth rates. We consider exogenous technical progress - Uzawa (1965) - Lucas (1988) technology would not imply significant changes for our purposes.

Admit, then, that the production function is CRS of the type:

$$F(K_{t-1}, A_{t-1} L_t) = A_{t-1} L_t F(\frac{k_{t-1}}{A_{t-1}}, 1) = A_{t-1} L_t f(\frac{k_{t-1}}{A_{t-1}}) \qquad (4.1)$$

$A_{t-1}$ is an efficiency factor affecting labor – labor-augmenting, Harrod-neutral technical progress -, exogenously growing at proportional rate x:

$$A_t = (1 + x) A_{t-1} \qquad (4.2)$$

Then one can convert (3.1) to:

$$(1 + n) \frac{k_t}{A_{t-1}} = (1 - d) \frac{k_{t-1}}{A_{t-1}} + f(\frac{k_{t-1}}{A_{t-1}}) - \frac{c_t}{A_{t-1}} -$$

$$- (g - h)\ dM_t\ f(\frac{k_{t-1}}{A_{t-1}}) / M_t + (1 - h)\ f(\frac{k_{t-2}}{A_{t-2}})\ dM_{t-1}/M_{t-1}\ (1 - d_h) / [(1 + x)(1 + n)] -$$

$$- rr\ \{f(\frac{k_{t-1}}{A_{t-1}}) - f(\frac{k_{t-2}}{A_{t-2}})\ (1 - d_r) / [(1 + n)(1 + x)]\} \qquad (4.3)$$

or

$$(1 + n)(1 + x)\frac{k_t}{A_t} = (1 - d)\frac{k_{t-1}}{A_{t-1}} + f(\frac{k_{t-1}}{A_{t-1}}) - \frac{c_t}{A_t}(1 + x) -$$

$$- (g - h)\ dM_t\ f(\frac{k_{t-1}}{A_{t-1}}) / M_t + (1 - h)\ f(\frac{k_{t-2}}{A_{t-2}})\ dM_{t-1}/M_{t-1}\ (1 - d_h) / [(1 + x)(1 + n)] -$$

$$- rr\ \{f(\frac{k_{t-1}}{A_{t-1}}) - f(\frac{k_{t-2}}{A_{t-2}})\ (1 - d_r) / [(1 + n)(1 + x)]\} \qquad (4.4)$$

If $U(c_t)$ is homogeneous of degree c, it will also be true that:

$$U(c_t) = A_t^{\,c}\, U(\frac{c_t}{A_t}) \qquad (4.5)$$

Then the problem is stated in such a way that $\frac{k_t}{A_t} = \hat{k}_t$ and $\frac{c_t}{A_t} = \hat{c}_t$ enjoy the same properties as $k_t$ and $c_t$ in the previous model – with also ρ replaced by $\hat{\rho} = \rho\,(1 + x)^c$, (1 + n) by $(1 + \hat{n}) = (1 + n)(1 + x)$ and the state equation with $\hat{c}_t$ multiplied by (1 + x): there will be a steady state *level* $\hat{k}^*$, $\hat{c}^*$ and m* that will be stable under similar requirements as before. In the balanced path, $y_t$, $k_t$[26] and $c_t$ will grow at rate x. The optimal rate of inflation will be $\pi^* = (1 + m^*)(1 + x)^{-1} - 1$. And of course, delays or losses associated with $dM_t / P_t$ and its lags suggest, again, an optimal long-run trend

[26] For Portugal, 1953-1995 – using information from Pinheiro *et al.*, (1997) -, the average annual per capita GDP real growth rate was 3.97% (of per capita real capital stock, 1953-1992 – using also data from Neves (1994) -, 4.36%) and not negligible.

of $m_t$ towards $m^* = 1 / (1 + n) - 1$, to a fixed aggregate money supply rule…

# 5. Extensions of MIU Modeling

## 5.1. "Taste for Real-Nominal Balance"

A felicity function valuing both real consumption and nominal (per capita) money balances, $U(c_t, M_t)$ increasing – possibly, concave or quasi-concave, and/or with decreasing first derivatives for SOC to hold - in both arguments, is able to produce a constant c* and M* along an optimal path, with reasonable assumptions, of $(c_t/M_t)$ in the presence of technical progress; then, the general price level, $P_t$, will approach stability in the long-run. The planner's problem becomes:

$$\underset{k_t, M_t}{Max} \sum_{t=1}^{\infty} \rho^t \; U\{(1 - d)\, k_{t-1} + f(k_{t-1}) - (1 + n)\, k_t - (g - h)\, [M_t - M_{t-1} / (1 + n)]\, f(k_{t-1}) / M_t + (1 - h)\, [(1 - d_h) / (1 + n)]\, [M_{t-1} - M_{t-2} / (1 + n)]\, f(k_{t-2}) / M_{t-1} - rr\, [f(k_{t-1}) - f(k_{t-2})\, (1 - d_r) / (1 + n)],\; M_t\} \qquad (5.1)$$

$c_t \geq 0$, $M_t \geq 0$, $M_t/M_{t-1} \geq f(k_{t-1}) / [(1 + n)\, f(k_{t-1}) + f(k_{t-2})]$

Given $k_{-1}$, $k_0$, $M_{-1}$, $M_0$(, $z_0$)

F.O.C., along with the restriction, require, for t = 1, 2, 3,…:

$$\frac{\partial W}{\partial k_t} = \rho^t ( - (1 + n) U_c(c_t, M_t) + \rho U_c(c_{t+1}, M_{t+1}) [(1 - d)+$$
$$f'(k_t) \{1 - rr - (g - h) [M_{t+1} - M_t / (1 + n)] / M_{t+1}\}] + \rho^2 U_c(c_{t+2}, M_{t+2}) f'(k_t) \{[(1 - d_h) / (1 + n)] (1 - h) [M_{t+1} - M_t / (1 + n)] / M_{t+1} + rr (1 - d_r) / (1 + n)\} ) = 0 \quad (5.2)$$

$$\frac{\partial W}{\partial M_t} = \rho^t ( \{ - (g - h) U_c(c_t, M_t) (M_{t-1} / M_t^2) f(k_{t-1}) + \rho$$
$$U_c(c_{t+1}, M_{t+1}) [(g - h) (1 / M_{t+1}) f(k_t) + (1 - h) (1 - d_h) (M_{t-1} / M_t^2) f(k_{t-1}) / (1 + n)] - \rho^2 U_c(c_{t+2}, M_{t+2}) (1 - h) (1 - d_h) (1 / M_{t+1}) f(k_t) / (1 + n) \} / (1 + n) + U_M(c_t, M_t) ) = 0 \quad (5.3)$$

The dynamics of the system can be stated in terms of $k_t$ and $M_t$, using the two FOC and the identity by which $c_t$ was replaced by the state equation. For convenience, we keep also $m_t = M_t / M_{t-1} - 1$ and write:

$$- (1 + n) U_c(c_t, M_t) + \rho U_c(c_{t+1}, M_{t+1}) [ (1 - d)+ f'(k_t) \{1 - rr - (g - h) [(1 + n) m_{t+1} + n] / [(1 + m_{t+1}) (1 + n)]\} ] + \rho^2 U_c(c_{t+2}, M_{t+2}) f'(k_t) \{ [(1 - d_h) / (1 + n)] (1 - h) [(1 + n) m_{t+1} + n] / [(1 + m_{t+1}) (1 + n)] + rr (1 - d_r) / (1 + n)\} = 0 \quad (5.4)$$

$$\{ - (g - h) U_c(c_t, M_t) f(k_{t-1}) / (1 + m_t) + \rho U_c(c_{t+1}, M_{t+1}) \{ (g - h) f(k_t) / (1 + m_{t+1}) + (1 - h) [(1 - d_h) / (1 + n)] f(k_{t-1}) / (1 + m_t) \} - \rho^2 U_c(c_{t+2}, M_{t+2}) (1 - h) [(1 - d_h) / (1 + n)] f(k_t) / (1 + m_{t+1}) \} / (1 + n) + M_t U_M (c_t, M_t) (1 + n) \} = 0 \quad (5.5)$$

or

$$f'(k_t) = \{(1 + n) U_c(c_t, M_t) - \rho U_c(c_{t+1}, M_{t+1}) (1 - d)\} / ( \rho U_c(c_{t+1}, M_{t+1}) \{1 - rr - (g - h) [(1 + n) m_{t+1} + n] / [(1 + m_{t+1}) (1 + n)]\} + + \rho^2 U_c(c_{t+2}, M_{t+2}) \{[(1 - d_h) / (1 + n)] (1 - h) [(1 + n) m_{t+1} + n] / [(1 + m_{t+1}) (1 + n)] + rr (1 - d_r) / (1 + n) \} ) \quad (5.6)$$

and

$(1 + m_t) / (1 + m_{t+1}) =$

$\{f(k_{t-1}) [(g – h) U_c(c_t, M_t) - \rho U_c(c_{t+1}, M_{t+1}) (1 – h) (1 – d_h) / (1 + n)] – (1 + n) U_M(c_t, M_t) M_t^2 / M_{t-1} \} /$

$\{f(k_t) [\rho U_c(c_{t+1}, M_{t+1}) (g – h) - \rho^2 U_c(c_{t+2}, M_{t+2}) (1 – h) (1 – d_h) / (1 + n)]\}$ (5.7)

The second equation is now active; there will be inventories, their build-up, involving expenditure leakage, is a price society pays for a stable value of currency… Out of steady state dynamics can be studied also embedding:

$c_t = (1 – d) k_{t-1} + f(k_{t-1}) - (1 + n) k_t – (g – h) \{1 – 1 / [(1 + n) (1 + m_t)]\} f(k_{t-1}) + (1 – h) [(1 – d_h) / (1 + n)] \{1 – 1 / [(1 + n) (1 + m_{t-1})]\} f(k_{t-2}) - rr [f(k_{t-1}) – f(k_{t-2}) (1 – d_r) / (1 + n)]$ (5.8)

Now, (5.6) and (5.7) would suggest a balanced path with a constant per capita capital stock, k*, zero growth of per capita money balances and zero inflation: m* = 0 – inventories will accumulate. Then:

$f'(k^*) = [(1 + n) - \rho (1 – d)] /$

$\{\rho [1 – rr - (g – h) n / (1 + n)] + \rho^2 \{[(1 – d_h) / (1 + n)] [(1 – h) n / (1 + n) ] + rr (1 – d_r) / (1 + n)\}$ (5.9)

and

$M^* = [(1 - \rho) / (1 + n)] \{(g – h) - \rho (1 – h) [(1 – d_h) / (1 + n)]\} [f(k^*) U_c(c^*, M^*)] / U_M(c^*, M^*)$ (5.10)

or

$P^* = [(1 - \rho) / (1 + n)] \{(g – h) - \rho (1 – h) [(1 – d_h) / (1 + n)]\} U_c(c^*, M^*) / U_M(c^*, M^*)$

As m* = 0 > 1 / (1 + n) – 1, (5.9) implies a smaller value of k* than (3.17). It increases with ρ, and now with h; it decreases with n, d, rr, $d_h$, and $d_r$ and now, with g. And it is independent of taste parameters.

If the marginat rate of substitution between M and c is constant – we may have a singular solution -, the general price level decreases with ρ, n, and h; it increases with g and $d_h$.

Also c* will then be smaller than implied under (3.18): now:

$$c^* = f(k^*) \{1 - rr [1 - (1 - d_r) / (1 + n)]\} - (n + d) k^* - f(k^*) [(g - h) - (1 - h) (1 - d_h) / (1 + n)] [1 - 1 / (1 + n)] \quad (5.11)$$

$c^* / y^*$, and therefore the savings rate, will be independent of taste parameters.

A (pseudo…) phase diagram representation of the per capita capital stock and money balances growth rate can be easily deducted for the case of a felicity function linear in consumption and additively separable in c and M, i.e., $U(c_t, M_t) = c_t + V(M_t)$ [27]. Then, $m_{t+1}$ can be inferred as a – negatively sloped, even if rr = 0 - function of $k_t$ only, $m_{t+1} = \phi(k_t)$ [28], from (5.6), depicted in Fig.1. For initial $k_0$, next period's growth rate of per capita money balances respond according to that function. As the system is stable, k will converge to k* over that function. (Yet, starting values of $k_{-1}$, $k_0$, $M_{-1}$ and $M_0$ may imply initial jumps of different direction than those of the light arrows depicted on the graph…)

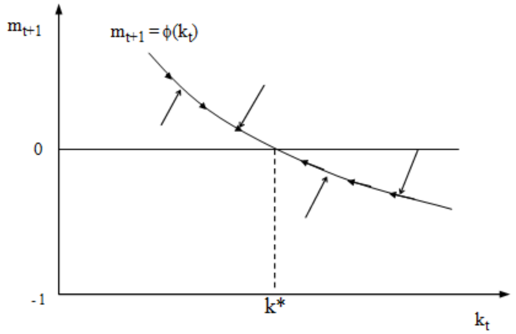


**Figure 1.**

[27] We found no reason now – as it occurs in the conventional Ramsey real growth model with linear felicity - for a necessary bang-bang solution for $c_t$… The presented solution would be, nevertheless, singular with respect to $c_t$ – and bang-bang paths (with zero consumption or consumption exhausting all capital net of inventories) may occur before it is reached…

[28] For Portugal, 1980-1993 – combining information from Pinheiro *et al.,* (1997) e Neves (1994) -, the correlation coefficient between per capita currency growth rate and lagged per capita capital stock was found negative, -0.58617, and highly significant (2.8%). Sign changes (yet significance disappears) if we consider a longer sample period – but, of course, neither per capita money balances nor prices have systematically declined (between 1953 and 1995, currency per capita rose at 11.52% per year – per capita M1 at 12.50% - and the GDP deflator at 9.74%)...

System dynamics can be defined between $M_t$ and $k_t$, using (5.7), over the optimal path:

$$(1 + m_t) / (1 + m_{t+1}) = [1 + \phi (k_{t-1})] / [1 + \phi (k_t)] = \{f(k_{t-1}) [(g - h) - \rho (1 - h) (1 - d_h) / (1 + n)] - (1 + n) V_M(M_t) M_t [1 + \phi (k_{t-1})]\} / \{f(k_t) [\rho (g - h) - \rho^2 (1 - h) (1 - d_h) / (1 + n)]\}$$

Yet, no single-time correspondence allows immediate graphical representation. If capital is low, one expects savings rate to be high – capital-labor ratio is increasing as well as productivity and real per capita money balances. Per capita nominal balances may be increasing but at a decreasing rate, because capital is low and reserve accumulation and loss from delay only increasingly (with product and capital) burdening, not faster than in the steady-state.

Using (1.3), and relying on $m_{t+1} = \phi (k_t)$ and the above, we can conclude that:

$$(1 + \pi_{t+1}) = (1 + m_{t+1}) f(k_{t-1}) / f(k_t) = [1 + \phi (k_t)] f(k_{t-1}) / f(k_t)$$

Inflation can move in the same or in opposite direction of $k_t$ and $y_t = f(k_{t-1})$ [29]. Developing the expression before last:

$$f(k_{t-1}) (1 + \pi_t) / [(1 + \pi_{t+1}) f(k_t)] = [1 + \phi (k_{t-1})] / [1 + \phi (k_t)] = \{f(k_{t-1}) [(g - h) - \rho (1 - h) (1 - d_h) / (1 + n)] - (1 + n) V_M(M_t) M_t [1 + \phi (k_{t-1})]\} / \{f(k_t) [\rho (g - h) - \rho^2 (1 - h) (1 - d_h) / (1 + n)]\}$$

A second possibility is to have taste for "real-nominal" balance: the consumer values the periodic real and nominal periodic consumption flow: $U(c_t, P_t c_t)$, with positive first derivatives. An alternative interpretation would be that the consumer has a felicity function $U'(c_t, 1 / P_t)$ that embeds distaste for $1 / P_t$, for a large real purchasing power of one nominal unit (for the real size of what

[29] The function $\phi (k_t)$, with $\phi'(k_t) < 0$, would predict a relation between output, $y_{t+1} = f(k_t)$, and per capita nominal growth that we would expect opposite to that of the Phillips curve. The same divergence cannot be inferred for the inflation path.

one, say, euro can buy…) – i.e., with a negative first derivative with respect to the second argument, $1 / P_t$. Then, the planner's problem is:

$$\underset{k_t, M_t}{Max} \sum_{t=1}^{\infty} \rho^t \; U\{(1 - d)\, k_{t-1} + f(k_{t-1}) - (1 + n)\, k_t - (g - h)\, [M_t - M_{t-1} / (1 + n)]\, f(k_{t-1}) / M_t + (1 - h)\, [(1 - d_h) / (1 + n)]\, [M_{t-1} - M_{t-2} / (1 + n)]\, f(k_{t-2}) / M_{t-1} - rr\, [f(k_{t-1}) - f(k_{t-2})\, (1 - d_r) / (1 + n)],\; [M_t / f(k_{t-1})]\, c(.)\} \quad (5.12)$$

$c_t \geq 0$, $M_t \geq 0$, $M_t/M_{t-1} \geq f(k_{t-1}) / [(1 + n)\, f(k_{t-1}) + f(k_{t-2})]$
Given $k_{-1}$, $k_0$, $M_{-1}$, $M_0$(, $z_0$)

c(.) denotes the same expression as included in the first argument. F.O.C., along with the restriction, require, for t = 1, 2, 3,…:

$$\frac{\partial W}{\partial k_t} = \rho^t \, (\, - (1 + n)\, [U_c(c_t, P_t\, c_t) + P_t\, U_{Pc}(c_t, P_t\, c_t)] + \rho\, [U_c(c_{t+1}, P_{t+1}\, c_{t+1}) + P_{t+1}\, U_{Pc}(c_{t+1}, P_{t+1}\, c_{t+1})]\, [(1 - d) + f'(k_t)\, \{1 - rr - (g - h)\, [M_{t+1} - M_t / (1 + n)] / M_{t+1}\}] + \rho^2\, [U_c(c_{t+2}, P_{t+2}\, c_{t+2}) + P_{t+2}\, U_{Pc}(c_{t+2}, P_{t+2}\, c_{t+2})]\, f'(k_t)\, \{[(1 - d_h) / (1 + n)]\, (1 - h)\, [M_{t+1} - M_t / (1 + n)] / M_{t+1} + rr\, (1 - d_r) / (1 + n)\} - \rho\, U_{Pc}(c_{t+1}, P_{t+1}\, c_{t+1})\, M_{t+1}\, c_{t+1}\, f'(k_t) / f(k_t)^2 \;) = 0 \quad (5.13)$$

$$\frac{\partial W}{\partial M_t} = \rho^t \, (\; \{ - (g - h)\, [U_c(c_t, P_t\, c_t) + P_t\, U_{Pc}(c_t, P_t\, c_t)]\, (M_{t-1} / M_t^2)\, f(k_{t-1}) + \rho\, [U_c(c_{t+1}, P_{t+1}\, c_{t+1}) + P_{t+1}\, U_{Pc}(c_{t+1}, P_{t+1}\, c_{t+1})]\, [(g - h)\, (1 / M_{t+1})\, f(k_t) + (1 - h)\, (1 - d_h)\, (M_{t-1} / M_t^2)\, f(k_{t-1}) / (1 + n)] - \rho^2\, [U_c(c_{t+2}, P_{t+2}\, c_{t+2}) + P_{t+2}\, U_{Pc}(c_{t+2}, P_{t+2}\, c_{t+2})]\, (1 - h)\, (1 - d_h)\, (1 / M_{t+1})\, f(k_t) / (1 + n)\, \} / (1 + n) + c_t\, U_{Pc}(c_t, P_t\, c_t) / f(k_{t-1})\; ) = 0 \quad (5.14)$$

The dynamics of the system can be stated in terms of $k_t$ and $M_t$, using the two FOC and the identity by which $c_t$ was replaced by the state equation. For convenience, we keep also $m_t = M_t / M_{t-1} - 1$ and write:

$$- (1 + n) [U_c(c_t, P_t c_t) + P_t U_{Pc}(c_t, P_t c_t)] + \rho [U_c(c_{t+1}, P_{t+1} c_{t+1}) + P_{t+1} U_{Pc}(c_{t+1}, P_{t+1} c_{t+1})] [ (1 - d)+ f'(k_t) \{1 - rr - (g - h) [(1 + n) m_{t+1} + n] / [(1 + m_{t+1}) (1 + n)]\} ] + \rho^2 [U_c(c_{t+2}, P_{t+2} c_{t+2}) + P_{t+2} U_{Pc}(c_{t+2}, P_{t+2} c_{t+2})] \; f'(k_t) \{[(1 - d_h) / (1 + n)] (1 - h) [(1 + n) m_{t+1} + n] / [(1 + m_{t+1}) (1 + n)] + rr (1 - d_r) / (1 + n)\} - \rho U_{Pc}(c_{t+1}, P_{t+1} c_{t+1}) P_{t+1} c_{t+1} f'(k_t) / f(k_t) = 0 \quad (5.15)$$

$$\{ - (g - h) [U_c(c_t, P_t c_t) + P_t U_{Pc}(c_t, P_t c_t)] \; f(k_{t-1}) / (1 + m_t) + \rho [U_c(c_{t+1}, P_{t+1} c_{t+1}) + P_{t+1} U_{Pc}(c_{t+1}, P_{t+1} c_{t+1})] \{ (g - h) f(k_t) / (1 + m_{t+1}) + (1 - h) [(1 - d_h) / (1 + n)] f(k_{t-1}) / (1 + m_t) \} - \rho^2 [U_c(c_{t+2}, P_{t+2} c_{t+2}) + P_{t+2} U_{Pc}(c_{t+2}, P_{t+2} c_{t+2})] (1 - h) [(1 - d_h) / (1 + n)] f(k_t) / (1 + m_{t+1}) \} / (1 + n) + M_t c_t U_{Pc}(c_t, P_t c_t) / f(k_{t-1}) = 0 \quad (5.16)$$

or

$$f'(k_t) = \{(1 + n) [U_c(c_t, P_t c_t) + P_t U_{Pc}(c_t, P_t c_t)] - \rho [U_c(c_{t+1}, P_{t+1} c_{t+1}) + P_{t+1} U_{Pc}(c_{t+1}, P_{t+1} c_{t+1})] (1 - d) \} /$$
$$( \rho [U_c(c_{t+1}, P_{t+1} c_{t+1}) + P_{t+1} U_{Pc}(c_{t+1}, P_{t+1} c_{t+1})] \{1 - rr - (g - h) [(1 + n) m_{t+1} + n] / [(1 + m_{t+1}) (1 + n)]\} +$$
$$+ \rho^2 [U_c(c_{t+2}, P_{t+2} c_{t+2}) + P_{t+2} U_{Pc}(c_{t+2}, P_{t+2} c_{t+2})] \; \{[(1 - d_h) / (1 + n)] (1 - h) [(1 + n) m_{t+1} + n] / [(1 + m_{t+1}) (1 + n)] + rr (1 - d_r) / (1 + n) \} - \rho U_{Pc}(c_{t+1}, P_{t+1} c_{t+1}) P_{t+1} c_{t+1} / f(k_t) \; ) \quad (5.17)$$

and

$$(1 + m_t) / (1 + m_{t+1}) =$$
$$( f(k_{t-1}) \{(g - h) [U_c(c_t, P_t c_t) + P_t U_{Pc}(c_t, P_t c_t)] - \rho [U_c(c_{t+1}, P_{t+1} c_{t+1}) + P_{t+1} U_{Pc}(c_{t+1}, P_{t+1} c_{t+1})] (1 - h) (1 - d_h) / (1 + n)\} - (1 + n) \; [c_t U_{Pc}(c_t, P_t c_t) / f(k_{t-1})] M_t^2 / M_{t-1} ) /$$
$$\{f(k_t) [\rho [U_c(c_{t+1}, P_{t+1} c_{t+1}) + P_{t+1} U_{Pc}(c_{t+1}, P_{t+1} c_{t+1})] (g - h) - \rho^2 [U_c(c_{t+2}, P_{t+2} c_{t+2}) + P_{t+2} U_{Pc}(c_{t+2}, P_{t+2} c_{t+2})] \; (1 - h) (1 - d_h) / (1 + n)]\} \quad (5.18)$$

Now, (5.17) and (5.18) would suggest a balanced path with a constant per capita capital stock, k*, and zero growth of per capita money balances and zero inflation: m* = 0. Then:

$$f'(k^*) = [(1 + n) - \rho\,(1 - d)\,] \,/$$
$$(\rho\,[1 - rr - (g - h)\,n / (1 + n)] + \rho^2\ \{[(1 - d_h) / (1 + n)]\,[(1 - h)\,n / (1 + n)\,] + rr\,(1 - d_r) / (1 + n)]\} - \rho\,\{P^*\,c^*\,U_{Pc}(c^*, P^*\,c^*) / [U_c(c^*, P^*\,c^*) + P^*\,U_{Pc}(c^*, P^*\,c^*)]\} / f(k^*)\,) \quad (5.19)$$

and

$$M^* = [(1 - \rho) / (1 + n)]\ \{(g - h) - \rho\,(1 - h)\,[(1 - d_h) / (1 + n)]\}\ f(k^*)^2\,[U_c(c^*, P^*\,c^*) + P^*\,U_{Pc}(c^*, P^*\,c^*)] / [c^*\,U_{Pc}(c^*, P^*\,c^*)\,]$$

or

$$P^* = [(1 - \rho) / (1 + n)]\ \{(g - h) - \rho\,(1 - h)\,[(1 - d_h) / (1 + n)]\}\ f(k^*)\,[U_c(c^*, P^*\,c^*) + P^*\,U_{Pc}(c^*, P^*\,c^*)] / [c^*\,U_{Pc}(c^*, P^*\,c^*)\,] \quad (5.20)$$

The expressions have, therefore similar structure as (5.9) and (5.10) – enjoying similar properties. (5.11) – as (5.8) – still hold.

## 5.2. "Taste for Nominal Growth"

We allow now for inflation rate itself – or deflation rate… - to enter the felicity function. Let $1/(1 + m_t) = m'$. Then, $U_{m'}(c, m') < 0$ – equivalent to $U_m[c, 1/(1 + m)] > 0$ - implies that individuals like nominal income growth – they "like inflation". A similar formulation would display consumers' felicity as a function $U'(c_t, P_{t-1} / P_t)$ that embeds distaste for *increases* (and not only levels as in the previous section) in $1 / P_{t-1}$, for increases in the real purchasing power of one nominal unit (of say, one euro…) – i.e., with a negative first derivative with respect to the second argument, $P_{t-1} / P_t$. The inclusion of $P_{t-1} / P_t$ in felicity (as one might argue for that of $1/P_t$ before) would capture preferences for, attitudes towards, unit of account stability.

The planner's problem is:

$$\underset{k_t, M_t}{Max} \sum_{t=1}^{\infty} \rho^t\ U\{(1 - d)\,k_{t-1} + f(k_{t-1}) - (1 + n)\,k_t - (g - h)\,[M_t - M_{t-1} / (1 + n)]\,f(k_{t-1}) / M_t + (1 - h)\,[(1 - d_h) / (1 + n)]\,[M_{t-1} - M_{t-}$$

$_2$ / (1 + n)] f($k_{t-2}$) /$M_{t-1}$ - rr [f($k_{t-1}$) – f($k_{t-2}$) (1 – $d_r$) / (1 + n)], $M_{t-1}$ / $M_t$} (5.21)

$c_t \geq 0$, $M_t \geq 0$, $M_t/M_{t-1} \geq f(k_{t-1}) / [(1 + n) f(k_{t-1}) + f(k_{t-2})]$

Given $k_{-1}$, $k_0$, $M_{-1}$, $M_0$(, $z_0$)

F.O.C., along with the restriction, require, for t = 1, 2, 3,…:

$$\frac{\partial W}{\partial k_t} = \rho^t ( - (1 + n) U_c[c_t, 1/(1 + m_t)] + \rho U_c[c_{t+1}, 1 / (1 + m_{t+1})] [(1 - d)+ f'(k_t) \{1 - rr - (g - h) [M_{t+1} - M_t / (1 + n)] /M_{t+1}\}] + \rho^2 U_c[c_{t+2}, 1 / (1 + m_{t+2})] f'(k_t) \{[(1 - d_h) / (1 + n)] (1 - h) [M_{t+1} - M_t / (1 + n)] / M_{t+1} + rr (1 - d_r) / (1 + n)\} ) = 0 \quad (5.22)$$

$$\frac{\partial W}{\partial M_t} = \rho^t ( \{ - (g - h) U_c[c_t, 1/(1 + m_t)] (M_{t-1} / M_t^2) f(k_{t-1}) + \rho U_c[c_{t+1}, 1 / (1 + m_{t+1})] \{(g - h) (1 / M_{t+1}) f(k_t) + (1 - h) (1 - d_h) (M_{t-1} / M_t^2) f(k_{t-1}) / (1 + n)\} - \rho^2 U_c[c_{t+2}, 1 / (1 + m_{t+2})] (1 - h) (1 - d_h) (1 / M_{t+1}) f(k_t) / (1 + n) \} / (1 + n) - U_m'(c_t, M_t/M_{t-1}) (M_{t-1} / M_t^2) + \rho U_m'(c_{t+1}, M_{t+1}/M_t) (1 / M_t) ) = 0 \quad (5.23)$$

The dynamics of the system can be stated in terms of $k_t$ and $m_t$ = $M_t$ / $M_{t-1}$ – 1 using the two FOC and the identity by which $c_t$ was replaced by the state equation as

$$- (1 + n) U_c[c_t, 1/(1 + m_t)] + \rho U_c[c_{t+1}, 1 / (1 + m_{t+1})] [ (1 - d)+ f'(k_t) \{1 - rr - (g - h) [(1 + n) m_{t+1} + n] / [(1 + m_{t+1}) (1 + n)]\} ] + \rho^2 U_c[c_{t+2}, 1 / (1 + m_{t+2})] f'(k_t) \{[(1 - d_h) / (1 + n)] (1 - h) [(1 + n) m_{t+1} + n] / [(1 + m_{t+1}) (1 + n)] + rr (1 - d_r) / (1 + n)\} ) = 0 \quad (5.24)$$

$\{ - (g - h) U_c[c_t, 1/(1 + m_t)] f(k_{t-1}) / (1 + m_t) + \rho U_c[c_{t+1}, 1 / (1 + m_{t+1})] \{ (g - h) f(k_t) / (1 + m_{t+1}) + (1 - h) [(1 - d_h) / (1 + n)] f(k_{t-1}) / (1 + m_t) \} - \rho^2 U_c[c_{t+2}, 1 / (1 + m_{t+2})] (1 - h) [(1 - d_h) /$

$(1 + n)] f(k_t) / (1 + m_{t+1}) \} / (1 + n) - U_m'[c_t, 1 / (1 + m_t)] / (1 + m_t) + \rho U_m'[c_{t+1}, 1 / (1 + m_{t+1})] \} = 0$ (5.25)

or

$$f'(k_t) = \{(1 + n) U_c[c_t, 1/(1 + m_t)] - \rho U_c[c_{t+1}, 1 / (1 + m_{t+1})] (1 - d)\} / \Big( \rho U_c[c_{t+1}, 1 / (1 + m_{t+1})] \{1 - rr - (g - h) [(1 + n) m_{t+1} + n] / [(1 + m_{t+1}) (1 + n)]\} + \rho^2 U_c[c_{t+2}, 1 / (1 + m_{t+2})] \{[(1 - d_h) / (1 + n)] (1 - h) [(1 + n) m_{t+1} + n] / [(1 + m_{t+1}) (1 + n)] + rr (1 - d_r) / (1 + n) \} \Big) \quad (5.26)$$

and

$$(1 + m_t) / (1 + m_{t+1}) = \Big(f(k_{t-1}) \{(g - h) U_c[c_t, 1/(1 + m_t)] - \rho U_c[c_{t+1}, 1 / (1 + m_{t+1})] (1 - h) (1 - d_h) / (1 + n) \} + U_m'[c_t, 1 / (1 + m_t)] \Big) / \Big( f(k_t) \{\rho U_c[c_{t+1}, 1 / (1 + m_{t+1})] (g - h) - \rho^2 U_c[c_{t+2}, 1 / (1 + m_{t+2})] (1 - h) (1 - d_h) / (1 + n) \} + \rho U_m'[c_{t+1}, 1 / (1 + m_{t+1})] \Big) \quad (5.27)$$

Out of steady state dynamics can be studied also embedding:

$$c_t = (1 - d) k_{t-1} + f(k_{t-1}) - (1 + n) k_t - (g - h) \{1 - 1 / [(1 + n) (1 + m_t)]\} f(k_{t-1}) + (1 - h) [(1 - d_h) / (1 + n)] \{1 - 1 / [(1 + n) (1 + m_{t-1})]\} f(k_{t-2}) - rr [f(k_{t-1}) - f(k_{t-2}) (1 - d_r) / (1 + n)] \quad (5.28)$$

From the system:

$$k_t = k[m_t, m_{t+1}, m_{t+2}, c(k_t, k_{t-1}, k_{t-2}, m_t, m_{t-1}), c(k_{t+1}, k_t, k_{t-1}, m_{t+1}, m_t), c(k_{t+2}, k_{t+1}, k_t, m_{t+1}, m_{t+2})]$$

$$m_t = m[m_t, m_{t+2}, m_{t+1}, c(k_t, k_{t-1}, k_{t-2}, m_t, m_{t-1}), c(k_{t+1}, k_t, k_{t-1}, m_{t+1}, m_t), c(k_{t+2}, k_{t+1}, k_t, m_{t+1}, m_{t+2})]$$

The equations entail forward-looking (with respect to $k_{t-2}$, $k_{t-1}$, $m_{t-2}$ and $m_{t-1}$) as backward-looking elements (with respect to $k_{t+1}$, $k_{t+2}$, $m_{t+1}$ and $m_{t+2}$). We could derive an autonomous system of autonomous equations in canonical forward-looking form – which would involve solving the system, and not only each equation, with respect to the highest leads of the two variables present:

$$k_{t+2} = k1(m_{t-1}, m_t, m_{t+1}, k_{t-2}, k_{t-1}, k_t, k_{t+1})$$
$$m_{t+2} = m1(m_{t-1}, m_t, m_{t+1}, k_{t-2}, k_{t-1}, k_t, k_{t+1})$$

It can be transformed in a system of seven first-order difference equations [30]. We could then inspect its eigenvalues [31] to inquire about stability. Being a higher than a 2x2 system, the task becomes cumbersome – and possibly generating many special cases; phase diagram analysis becomes blurred.

(5.27) would suggest – again – a balanced deceleration such that $(1 + mm^*) = (1 + m_{t+1}) / (1 + m_t) = \rho$. However, steady-state values of m*, k* and c* are compatible with (5.22) – and (5.23) -, requiring:

$$f'(k^*) = [(1 + n) - \rho (1 - d)] / \Big( \rho \{1 - rr - (g - h) [(1 + n) m^* + n] / [(1 + m^*) (1 + n)]\} + \rho^2 \{[(1 - d_h) / (1 + n)] (1 - h) [(1 + n) m^* + n] / [(1 + m^*) (1 + n)] + rr (1 - d_r) / (1 + n) \} \Big) \quad (5.29)$$

and

$$\{(g - h) - \rho (1 - h) [(1 - d_h) / (1 + n)]\} f(k^*) = - (1 + n) U_m'[c^*, 1/(1+m^*)] / U_c[c^*, 1/(1+m^*)] \quad (5.30)$$

In general, the steady-state (1 + m*) is expected to be compatible with the inequality constraints, which therefore lose their active role. Also, *a priori*, m* is unrestricted: it can be positive or negative, depending on the shape of U(c, m'). Yet, an interior solution generating a steady-state value of m* lower than 1 / (1 + n) – 1 could imply a reversion to the path of the standard felicity function, U(c) – i.e., of section 3.1. Then, for m* > 1 / (1 + n) – 1, (5.29) implies a smaller value of k* than (3.17). Also c* will then be smaller than implied under (3.18): now:

$$c^* = f(k^*) \{1 - rr [1 - (1 - d_r) / (1 + n)]\} - (n + d) k^* - f(k^*) [(g - h) - (1 - h) (1 - d_h) / (1 + n)] \{1 - 1 / [(1 + m^*) (1 + n)]\} \quad (5.31)$$

[30] See Azariadis (1993), p. 6.
[31] See Azariadis (1993), p. 59.

g > 1 and 1 / [(1 + m*) (1 + n)] < 1 imply that the last term is negative. As f'(k*) {1 - rr [1 – (1 – $d_r$) / (1 + n)]} > n + d (easily proven for rr = 0), its effect adds to that of the already smaller k*.

One could wonder what possible features would U[c, 1/(1+m)] exhibit. A possibility – that would not contend with balanced growth steady-states in the presence of the usual exogenous technical progress – would include general functions of the product of powers of the two arguments $U(c_t, M_{t-1}/M_t) = U[c_t^{\,a}\left(\frac{M_{t-1}}{M_t}\right)^{-b}]$. Then the marginal rate of transformation between the two, $-U_m'[c_t, 1/(1+m_t)] / U_c[c_t, 1/(1+m_t)] = (b/a)\, c_t\, (1 + m_t)$, convenient for the second equation.

Likewise, $U[c_t^{\,a} \exp\left(-b\frac{M_{t-1}}{M_t}\right)]$ implies $-U_m'[c_t, 1/(1+m_t)] / U_c[c_t, 1/(1+m_t)] = (b/a)\, c_t$.

An alternative would extend the commonly used constant intertemporal elasticity of substitution felicity function to incorporate $M_t/M_{t-1}$ at the elasticity itself. Say:

$$U(c_t, M_{t-1}/M_t) = \frac{c_t^{\left(a-\frac{1}{\sigma}\frac{M_{t-1}}{M_t}\right)}}{a-\frac{1}{\sigma}\frac{M_{t-1}}{M_t}} \qquad (5.32)$$

Then, $U_c(c_t, M_{t-1}/M_t) = c_t^{\left(a-1-\frac{1}{\sigma}\frac{M_{t-1}}{M_t}\right)}$ and $U_m'(c_t, M_{t-1}/M_t)$

$$= \frac{\left[1-\left(a-\frac{1}{\sigma}\frac{M_{t-1}}{M_t}\right)\log(c_t)\right] c_t^{\left(a-\frac{1}{\sigma}\frac{M_{t-1}}{M_t}\right)}}{\sigma\left(a-\frac{1}{\sigma}\frac{M_{t-1}}{M_t}\right)^2}.$$

With inflation –in – utility, the problem would not change much. Moreover, we could argue that if price increases were the driving motive, their effect should nevertheless be weighted by the appropriate quantity index – and we would thus recover the nominal balance increase formulation.

In a steady-state of an economy with only a storable good, per capita money balances with taste for inflation should grow at the same rate as the general price level and at rate $\pi^* = m^*$. And

$$k_Z^* = (f - c^*) / (n + d) + f \{(1 - h) (1 - rr) (1 - d_h) / (1 + n) - [1 - h (1 - rr)]\} [m^* + n /(1 + n)] / [(n + d) (1 + m^*)] \tag{5.33}$$

A (completely different…) alternative rationalization of inflation in utility would justify its introduction to represent hypothetical non-monetized consumption of part of inventory stocks – as charity, take-home goods: if full changes in inventories are not consumed, partial consumption of it would be observed and be aligned with $dM_t/P_t$; as worse products would be left in warehouses, the utility derived from this consumption differs from that of $c_t$, and a differential inclusion in felicity, that would take the form $U(c_t, dM_t/P_t) = U\{c_t, f(k_{t-1}) - f(k_{t-2}) P_{t-1} / [P_t (1 + n)]\}$, would be justified. Hypothetically, those inventories could be sold at lower than market prices – in an attempt to release them "in low season", in sales; then, paradoxically, the pursuit of a good bargain – of lower prices and consumption of such items … - by consumers would lead the economy to inflation…

A generalization that would allow for full inventory change recovery in this fashion would allow for a felicity $U\{c_t, z_t - z_{t-1} [(1 - d_h) / (1 + n)]\}$.

Nevertheless, such interpretations of the meaning of inflation in utility would inspire us again to suggest the convenience of an additional (now a third) argument in felicity, a nominal per capita aggregate – nominal per capita money balances, or nominal per capita consumption -, to proxy taste for price – unit account real size - stability.

Overall, the welfare maximizer central planner facing inflation-in-utility appears compatible with a central monetary authority oriented by an ad-hoc objective function with inflation and output as arguments - as sometimes assumed in economic research.

## 5.3. Discounted Nominal Utility

An alternative formulation would consider that the appropriate felicity function would have nominal consumption as argument – having or not $M_t$ as a second argument - consistently, being discounted by a “nominal” discount factor, $(\rho \frac{P_{t-1}}{P_t})$ for period t [32],

so that individuals maximize:

$$\text{Max} \sum_{t=1}^{\infty} \rho^t \prod_{s=1}^{t} \left( \frac{P_{s-1}}{P_s} \right) U(P_t\, c_t)^{33} = \sum_{t=1}^{\infty} \rho^t \frac{P_0}{P_t} U(P_t\, c_t) \quad (5.34)$$

After the replacement of the transactions money demand constraint and dividing by $P_0$:

$$\underset{k_t, M_t}{Max} \sum_{t=1}^{\infty} \rho^t \frac{f(k_{t-1})}{M_t} U\left(\frac{M_t}{f(k_{t-1})} c_t\right)$$

or $\underset{k_t, M_t}{Max} \sum_{t=1}^{\infty} \rho^t \frac{f(k_{t-1})}{M_t} U\left(\frac{M_t}{f(k_{t-1})} \{(1-d) k_{t-1} + f(k_{t-1}) - (1+n) k_t - (g-h) [M_t - M_{t-1} / (1+n)] f(k_{t-1}) / M_t + (1-h) [(1-d_h) / (1+n)] [M_{t-1} - M_{t-2} / (1+n)] f(k_{t-2}) / M_{t-1} - rr [f(k_{t-1}) - f(k_{t-2}) (1-d_r) / (1+n)]\}\right)$ (5.35)

Given $k_{-1}$, $k_0$, $M_{-1}$, $M_0$(, $z_0$)

If $\frac{U(P_t c_t)}{P_t}$ increases with $P_t$ – i.e., whenever $U_{pc}(P_t c_t) P_t c_t > U(P_t c_t)$, the elasticity of U(.) with respect to the argument is larger than 1 - implies that individuals prefer small real size of the nominal unit, $\frac{1}{P_t}$. F.O.C., along with the restriction, require, for t = 1, 2, 3,…:

[32] ρ is still a real discount factor…

[33] If we used $\sum_{t=1}^{\infty} \rho^t \left( \frac{P_{t-1}}{P_t} \right)^t$ U($P_t$ $c_t$) instead, we would generate time inconsistency – FOC would depend on t. (As is well known, the comment also applies in the presence of variable discount factors if use the power of the period’s discount factor instead of the factored product of all previous ones to discount each term…)

$$\frac{\partial W}{\partial k_t} = \rho^t \; ( - (1 + n) U_{pc}(P_t c_t) + \rho \, U_{pc}(P_{t+1} c_{t+1}) \, [(1 - d)+$$

$f'(k_t) \{1 - rr - (g - h) [M_{t+1} - M_t / (1 + n)] / M_{t+1}\}] + \rho^2 U_{pc}(P_{t+2} c_{t+2}) f'(k_t) \{[(1 - d_h) / (1 + n)] (1 - h) [M_{t+1} - M_t / (1 + n)] / M_{t+1} + rr (1 - d_r) / (1 + n)\} + \rho f'(k_t) [U(P_{t+1} c_{t+1}) / M_{t+1}$

$$- \frac{M_{t+1}}{f(k_t)^2} P_{t+1} c_{t+1} U_{pc}(P_{t+1} c_{t+1})] \;) = 0 \qquad (5.36)$$

$$\frac{\partial W}{\partial M_t} = \rho^t \; ( \; \{ - (g - h) U_{pc}(P_t c_t) (M_{t-1} / M_t^2) f(k_{t-1}) + \rho$$

$U_{pc}(P_{t+1} c_{t+1}) [(g - h) (1 / M_{t+1}) f(k_t) + (1 - h) (1 - d_h) (M_{t-1} / M_t^2) f(k_{t-1}) / (1 + n)] - \rho^2 U_{pc}(P_{t+2} c_{t+2}) (1 - h) (1 - d_h) (1 / M_{t+1}) f(k_t) / (1 + n) \} / (1 + n) + [c_t U_{pc}(P_t c_t) / M_t - U(P_t c_t)$

$$f(k_{t-1}) / M_t^2] \;) = 0 \qquad (5.37)$$

A steady-state might be possible.

If $U(x) = A x^c$, $c > 0$ and A constant, as $\frac{U(P_t c_t)}{P_t} = A P_t^{(c-1)} c_t^c$ and the typical term of the welfare function $\rho^t P_t^{(c-1)} A c_t^c$, $P_t^{(c-1)} \rho^t$ could tend to a constant along the optimal path, i.e., $P_t^{(c-1)} / P_{t-1}^{(c-1)}$ would tend to $1 / \rho$; then, $(P_t / P_{t-1})^* = (M_t / M_{t-1})^* = 1 + m^* = \rho^{1/(1-c)}$ (provided, $\rho^{1/(1-c)} > 1 / (1 + n)$). Then, if $c < 1$, $m^* < 0$: if the intertemporal elasticity of substitution $(1/(1 - c))$ is larger than 1, per capita money balances and price level decrease and at a faster pace than the Friedman's rule ($\rho^{1/(1-c)} < \rho < 1$); if $c > 1$ (provided SOC still hold…), $m^*$ will be positive – and also steady-state inflation.

# 6. Nominal Growth Productivity Effects

With nominal growth –in – production - say, making inventories to be sold more easily; or (unmodelled) intermediate products cheaper:

$\underset{k_t, M_t}{Max} \sum_{t=1}^{\infty} \rho^t$ U{(1 – d) $k_{t-1}$ + f($k_{t-1}$, $M_t/M_{t-1}$) - (1 + n) $k_t$ – (g – h) [$M_t$ – $M_{t-1}$ / (1 + n)] f($k_{t-1}$, $M_t/M_{t-1}$) /$M_t$ + (1 – h) [(1 – $d_h$) / (1 + n)] [$M_{t-1}$ – $M_{t-2}$ / (1 + n)] f($k_{t-2}$, $M_{t-1}/M_{t-2}$)/$M_{t-1}$ - rr [f($k_{t-1}$) – f($k_{t-2}$) (1 – $d_r$) / (1 + n)], $M_{t-1}$ / $M_t$} (6.1)

$c_t \geq 0$, $M_t \geq 0$, $M_t/M_{t-1} \geq f(k_{t-1}) / [(1 + n) f(k_{t-1}) + f(k_{t-2})]$
Given $k_{-1}$, $k_0$, $M_{-1}$, $M_0$(, $z_0$)

The F.O.C., along with the restriction, require, for t = 1, 2, 3,…:

$\frac{\partial W}{\partial k_t}$ = $\rho^t$ ( - (1 + n) $U_c(c_t)$ + ρ $U_c(c_{t+1})$ [(1 – d)+ $f_k$($k_t$, $M_{t+1}/M_t$) {1 – [1 – h (1 - rr)] [$M_{t+1}$ – $M_t$ / (1 + n)] /$M_{t+1}$}] + $\rho^2$ $U_c(c_{t+2})$ {(1 – h) [(1 – $d_h$) / (1 + n)] [$M_{t+1}$ – $M_t$ / (1 + n)] $M_{t+1}$ + rr (1 – $d_r$) / (1 + n)} $f_k$($k_t$, $M_{t+1}/M_t$) / = 0 (6.2)

$$\frac{\partial W}{\partial M_t} = \rho^t \{ \quad ( - [1 - h\ (1 - rr)]\ U_c(c_t)\ (M_{t-1} / M_t^2)\ f(k_{t-1},$$

$M_t/M_{t-1}) + \rho\ U_c(c_{t+1})\ \{[1 - h\ (1 - rr)]\ (1 / M_{t+1})\ f(k_t, M_{t+1}/M_t)$

$+ (1 - h)\ (1 - rr)\ [(1 - d_h) / (1 + n)]\ (M_{t-1} / M_t^2)\ f(k_{t-1}, M_t/M_{t-1})\}$

$- \rho^2\ U_c(c_{t+2})\ (1 - h)\ (1 - rr)\ [(1 - d_h) / (1 + n)]\ (1 / M_{t+1})\ f(k_t,$
$M_{t+1}/M_t) +$

$+ U_c(c_t)\ \{1 - [1 - h\ (1 - rr)]\ [M_t - M_{t-1} / (1 + n)]\ /M_t\}\ f_m(k_{t-1},$
$M_t/M_{t-1})\ /M_{t-1} \quad ) / (1 + n) +$

$- \rho\ U_c(c_{t+1})\ f_m(k_t, M_{t+1}/M_t)\ \{1 - [1 - h\ (1 - rr)]\ [M_{t+1} - M_t /$
$(1 + n)]\ /M_{t+1}\}\ (M_{t+1} / M_t^2) +$

$+ \rho\ U_c(c_{t+1})\ (1 - h)\ (1 - rr)\ [(1 - d_h)/(1 + n)]\ [M_{t-1} - M_{t-2} / (1$
$+ n)]\ f_m(k_{t-1}, M_t/M_{t-1})/M_t\ (1 / M_{t-1})$

$- \rho^2\ U_c(c_{t+2})\ (1 - h)\ (1 - rr)\ [(1 - d_h) / (1 + n)]\ [M_{t+1} - M_t / (1$
$+ n)]\ f_m(k_t, M_{t+1}/M_t) / M_{t+1}\ (M_{t+1} / M_t^2)\ \} = 0$ (6.3)

It is immediate to recognize a solution pattern similar to that of section 5.2. Provided $m^* > 1 / (1 + n) - 1$ – the opposite would imply that the inventory equation would become binding somewhen… - a long-run trend with some inflation – or deflation other than implied by fixed aggregate money supply – may result.

As with felicity, the direct use of unsold inventories in the own firm production process – as inferior capital goods - could justify the inclusion of similar terms – say, inflation itself – as a second argument of (now) the average product function, for example, admit $f(k_{t-1}, 1 - P_{t-1}/P_t)$ (with a positive partial derivative with respect to the second argument, implying a positive impact of inflation). $f[k_{t-1}, z_t - z_{t-1} (1 - d_h) / (1 + n)]$ could represent the full use of change in inventories as an intermediate product. $f(k_{t-1}, z_t)$ would be another interesting possibility, with $z_t$ as a "differentiated" capital factor…

The inclusion of the conversion mechanism and Clower's delay assumption provide an accountingly consistent argument generating inventory build-up. Rationales as those of the previous paragraph – or those for inflation in utility forwarded at the end of sub-section 5.2 – would suggest ways in which the economy would circumvent, at least in part, the implied losses. Implicitly, they entail some surpassing of the need for full expenditure monetization – even if not that of income, if (1.2) is maintained…

# 7. Equilibrium: Wages, Rental Prices, and Interest Rates

## 7.1. (Inefficiency of the) Competitive Equilibrium

Let us briefly outline the possible outcomes of a decentralized economy with a central authority with which the public also can exchange goods for money (and vice versa...). The underlying real world is that of section 3.1. Assume firms are instantaneous (or rather, periodic…) and individuals own factors that rent to them. Let $W_t$ denote the wage rate and $R_t$ the rental price of capital paid at time t. If they are paid in advance, due to the CRS assumption, marginal product factor pricing, $R_t = P_t\ f'(k_{t-1})$ and $W_t = P_t\ [f(k_{t-1}) - k_{t-1}\ f'(k_{t-1}) / (1 + n)]$, guarantees:

$$P_t\ L_t\ f(k_{t-1}) \ = \ W_t\ L_t + R_t\ K_{t-1} \qquad (7.1)$$

Firms will make no profits. Hence, any money management costs will be borne by individual consumers/investors. Individuals trade money for goods and goods for money in the economy; then they are also bound to do it with the central authority to avoid frauds… Because to obtain money they must have capital to collateralize borrowing – provide mortgages -, and then they have to pay interest when they do, it is as if they sell capital to the central bank for the money demanded. Loans, money loans, always pay interest – there is no way to distinguish net borrowers from individuals requesting intermediation… The real interest rate in the economy – the one at which individuals – or rather, households - discount consumption, or are willing to trade t's consumption/real goods for t+1 ones and therefore ask for to lend – is the marginal rate of substitution between ($L_{t+1}$ $c_{t+1}$) and ($L_t$ $c_t$) over the individuals welfare function minus 1 [34]:

$$- \left[\frac{d(L_{t+1}c_{t+1})}{d(L_t c_t)}\right]_{|\overline{W}} \text{ - } 1 \;=\; - \frac{L_{t+1}}{L_t}\left(\frac{dc_{t+1}}{dc_t}\right)_{|\overline{W}} \text{ -1 } \;=\; (1 + \text{n})$$

$$\left(\frac{\dfrac{\partial W}{\partial c_t}}{\dfrac{\partial W}{\partial c_{t+1}}}\right) \text{ - 1 } = (1+\text{n}) \frac{\rho^t U_c(c_t)}{\rho^{t+1} U_c(c_{t+1})} \text{ - 1 } = \frac{1+n}{\rho}\,\frac{U_c(c_t)}{U_c(c_{t+1})} \text{ - 1}$$

$$= \frac{1}{\rho'}\,\frac{U_c(c_t)}{U_c(c_{t+1})} \text{ - 1 } = r_t \text{ [35]} \qquad (7.2)$$

[34] In the Ramsey's model, the real rate of return to savings is – equated to - $r_t$ = f'($k_{t-1}$) – d = $R_t/P_t$ – d – see Barro & Sala-i-Martin (1995), p. 63-69. The term (1 – d) ($P_{k,t}$ – $P_{k,t-1}$) / $P_{k,t-1}$ should be added – see footnote 11. p. 69 of the same reference - when $P_{k,t}$ is the price of capital in units of consumables – in the one-sector model, fixed to 1.

[35] Notice that discount factors variable in time, say the maximand is $\sum_{t=1}^{\infty} \prod_{j=1}^{t}$ □$_j$ U($c_t$), would equate $\frac{1+n}{\rho_{t+1}}\,\frac{U_c(c_t)}{U_c(c_{t+1})}$ - 1 = $\frac{1}{\rho'_{t+1}}\,\frac{U_c(c_t)}{U_c(c_{t+1})}$ - 1 = $r_t$.

The solution for the conventional Ramsey model would require $\frac{U_c(c_t)}{U_c(c_{t+1})}$ $\frac{1}{\rho'_{t+1}}$ = (1 – d) + f'($k_t$); a steady state for c or k would hardly exist once the state equation (1 + n) $k_t$ = (1 – d) $k_{t-1}$ + f($k_t$) - $c_t$ must also be complied with and $\frac{U_c[-(1+\text{n})\,\text{k}_t + (1\ \text{d})\,\text{k}_{t\text{-}1} + \text{f}(\text{k}_t)]}{U_c[-(1+\text{n})\,\text{k}_{t+1} + (1\ \text{d})\,\text{k}_t + \text{f}(\text{k}_{t+1})]}\,\frac{1}{\rho'_{t+1}}$ = (1 – d) + f'($k_t$).

The problem here is not pure time consistency, but of a characterization – existence - of steady-state solution.

where ρ was replaced by $\rho = \rho' (1 + n)$ and ρ' denotes the individuals' discount factor when future household members are valued - $r_t$ differs from and implies the time structure of interest rates). Then, we can collapse the private system, assume it behaves as a price-taker towards the money aggregate that must meet $M_t = P_t f(k_{t-1})$ – price-takers towards $M_t$, facing exogenous $1 / P_t$…

The individuals recognize money operating costs, but the re-insertion is inadequately apprehended. They recognize taxes – reproducing the real reserve creation costs net of re-insertion "profits" of the central authority -, $T_t$, the sequence of which the government pre-announces, but those are taken as exogenous by the private sector.

$$\underset{c_t, k_t}{Max} \sum_{t=1}^{\infty} \rho^t \ U(c_t)$$

s.t: $(1 + n) \ k_t = (1 - d) k_{t-1} + f(k_{t-1}) - c_t - g \{f(k_{t-1}) - f(k_{t-2}) P_{t-1}/[P_t (1 + n)]\} - T_t$ (7.3)

or

$$\underset{k_t}{Max} \sum_{t=1}^{\infty} \rho^t \ U( (1 - d) k_{t-1} + f(k_{t-1}) - (1 + n) \ k_t - g \{f(k_{t-1}) - f(k_{t-2}) P_{t-1}/[P_t (1 + n)]\} - T_t ) \quad (7.4)$$

The inventory state equation does not necessarily restrain the private sector – its dynamics are dictated by the, exogenous from the private sector perspective, inflation rate and re-insertion, and we assume the former is not smaller than $1 / (n + 1) - 1$…

(If nominal money balances were also valued, and the individuals' felicity function of form $U(c_t, M_t)$, the *ex ante* money demand equation, $M_t = P_t f(k_{t-1})$, would be internalized – i.e., we could replace the felicity in the problem above by $U[c_t, P_t f(k_{t-1})]$ to derive the private sector's FOC.)

The central bank, through head taxes, then requests taxes:

$$\begin{aligned} T_t &= rr [f(k_{t-1}) - f(k_{t-2}) (1 - d_r) / (1 + n)] - db_t = \\ &= rr [f(k_{t-1}) - f(k_{t-2}) (1 - d_r) / (1 + n)] - h \{f(k_{t-1}) - f(k_{t-2}) P_{t-1}/[P_t (1 + n)]\} - (1 - h) \{f(k_{t-2}) - f(k_{t-3}) P_{t-2}/[P_{t-1} (1 + n)]\} (1 - d_h) / (1 + n) \end{aligned} \quad (7.5)$$

One can think that the bank rather requests $T'_t = T_t + [M_t / P_t - (M_{t-1} / P_{t-1}) (1 - d) / (1 + n)]$ of taxes (net of increase in real value of outstanding loans with the central bank) and announces a real value increase of $[M_t / P_t - (M_{t-1} / P_{t-1}) (1 - d) / (1 + n)]$ - so that the individuals know that society – they… - additionally owns that capital, $M_t / P_t$, - real backup of the paper people carry around, entrusted to the central authority - that is included in the one used in production; transfers of the pertaining profits, are internalized by individuals, and total capital $k_t$ enters their state equation. That would make eventual negative changes of money supply appear more immediate, not *ex-ante* dependent on taxes, specially if rr = 0. (We should not have $T' = T + (1 - rr) [M_t / P_t - (M_{t-1} / P_{t-1}) (1 - d) / (1 + n)]$ if money was issued against loans: then interest ends up being paid who knows whom...) The overall outcome is that of inserting (7.5) in (7.3) and (7.4) – and in conditional demands.

FOC for problem (7.4) generate:

$$\frac{\partial W}{\partial k_t} = \rho^t \{ - (1 + n) U_c(c_t) + \rho U_c(c_{t+1}) [(1 - d) + f'(k_t) - g f'(k_t)] + \rho^2 U_c(c_{t+2}) g f'(k_t) P_{t-1}/[P_t (1 + n)]\} = 0 \quad (7.6)$$

Or

$$f'(k_t) = \{[(1 + n) / \rho] U_c(c_t) - (1 - d) U_c(c_{t+1})\} / \{U_c(c_{t+1}) (1 - g) + \rho U_c(c_{t+2}) g P_{t-1}/[P_t (1 + n)] \quad (7.7)$$

With the state equation, they allow for implicit relations between the $k_t$'s answering to sequences of $P_t$'s, $k_t = k(P_t, P_{t+1}, P_{t+2},\ldots, T_t, T_{t+1}, T_{t+2},\ldots)$, and therefore, also of $y_t$'s. They imply money demands $M_t^d = P_t\, y(P_t, P_{t+1}, P_{t+2},\ldots T_t, T_{t+1}, T_{t+2}\ldots)$. By supplying – fixing – the $M_t$'s (by canceling or conceding money loans), the central authority determines prices. Consumption will obey – after replacement - (3.1).

If a steady-state is going emerge, condition (7.7) implies:

$$f'(k_{eq}^*) = \{[(1 + n) / \rho] - (1 - d)\} / [(1 - g) + \rho g (P_{t-1}/P_t)^* / (1 + n)] \quad (7.8)$$

But now, if the money authority can but fix $M_t$ - and $P_t$ – through (7.8), it can never achieve the first best of 3.1: the optimal inflation rate, that guarantees equality between (7.8) and (3.17), the optimal one k* that obeys:

$$f'(k^*) = [(1+n) - \rho\,(1-d)] / \{\rho\,(1-rr) + \rho^2\,[rr\,(1-d_r)/(1+n)]\} \qquad (7.9)$$

is not 1 / (1 + n) [36]… And as

$$z_t = z_{t-1}\,(1-d)/(1+n) + (g-h)\,f(k^*)\,[M_t - M_{t-1}/(1+n)]/M_t - (1-h)\,[(1-d_h)/(1+n)]\,f(k^*)\,[M_{t-1} - M_{t-2}/(1+n)]/M_{t-1} \qquad (7.10)$$

if $g > 1$, $M_t / M_{t-1} > 1 / (1 + n)$ – if $rr = 0$, the individual's discount rate (accounting for future generations) had to be smaller than population growth for it to be possible (n could not be zero if $1/\rho = 1 / [\rho' (1 + n)] > 1$…) - would imply inventories.

If $g = 1$, any path that started with some inventories and out of the steady-state might be impossible or non-optimal...

Nor in nor out of the steady-state can the two optimality conditions be compatible.

Then, the central authority could therefore fix $(P_t/P_{t-1})^* = 1 / (1 + n)$, and, because then:

$$f'(k_{eq}^*) = \{[(1+n)/\rho] - (1-d)\} / [(1-g) + \rho\, g] \qquad (7.11)$$

implying $k_{eq}^* < k^*$ if $(1 - g) + \rho\, g < (1 - rr) + \rho\ [rr\,(1 - d_r) / (1 + n)]$: subsidize capital services if $g\,(1 - \rho) > rr\,\{1 - \rho\ (1 - d_r) / (1 + n)\}$ – i.e., grant a subsidy s to firms per unit of profit, so that they equate $(1 - s)\, R_t\, K_t$ to $f'(k_{t-1})$ and extract that fiscal revenue from them in a lump-sum fashion (or capital owners' income, adding the implicit per capita revenue to $T_t$…); otherwise, tax them. (If $g > 1$, one would expect $k_{eq}^* < k^*$.)

Notice that individuals do not internalize the induced zero inventory balance policy – inventories are outside their control,

[36] The model does not imply price indeterminacy – see Dalziel (2000) for an historical discussion on the subject -, but sub-optimality of an "almost" free banking system…

only perceived by the central authority. Therefore, we cannot presume, when the monetary authority targets $m^* = 1 / (1 + n) - 1$, a steady-state where the private sector acts as if $g = 0$ and $f'(k_{eq}^*) = [(1 + n) / \rho] - (1 - d)$ (which in any case would not be optimal in the presence of official reserves.)

If there were no reserve requirement – after all, the central authority owns capital or lent money requiring mortgages -, $(P_t/P_{t-1})^* = \rho / (1 + n)$, or rather – replacing ρ by $\rho = \rho' (1 + n)$ to value future generations - $(P_t/P_{t-1})^* = \rho' (\rho' (1 + x)^c$ if there was exogenous technical progress and the utility function is homogeneous of degree c in the argument), the Friedman rule [37], would generate the optimal $f'(k^*)$ but would not guarantee zero inventories or the optimal consumption, not even if $g = 1$, which induce the same problems as before.

The central authority could therefore fix $(P_t/P_{t-1})^* = 1 / (1 + n)$, and, because:

$$f'(k_{eq}^*) = \{[(1 + n) / \rho] - (1 - d)\} / [(1 - g) + \rho g] > \{[(1 + n) / \rho] - (1 - d)\} \quad (7.12)$$

implying $k_{eq}^* < k^*$, subsidize capital services.

Eventually, in an alternative world: cash in the economy is deposited in commercial banks and all cash payments are made through bank transfers – yet cash must be there in advance. Then, whoever kept the money from period t to t-1 is known. This would allow the central authority to follow the efficient money stock policy and:

- obtain loans from the general (private…) public, paying the current nominal interest rate, to finance reserve formation – or buy them…

- pay to whomever proves he held cash balances from t-1 to t, approximately $(1 - rr)\, r_t'$ per nominal unit – $r_t'$ denoting a nominal interest rate -, transferring the implicit capital revenues it obtained net of depreciation, $(1 - rr) \{R_t (M_{t-1} L_{t-1}) / P_{t-1} + (M_{t-1} L_{t-1}) [(1 - d) / P_{t-1} - 1 / P_t]\} - rr\, d_r M_{t-1} L_{t-1}) / P_{t-1}$ in real terms, to those

[37] A zero nominal rate of interest.

who would have owned the capital from t-1 to t if the economy was a barter – deducting the official reserves loss [38]…

A slightly different allocation of money management costs would assign net losses $(g + 1) \{f(k_{t-1}) – f(k_{t-2}) P_{t-1}/[P_t (1 + n)]\} – ( h \{f(k_{t-1}) – f(k_{t-2}) P_{t-1}/[P_t (1 + n)]\} - (1 – h) \{f(k_{t-2}) – f(k_{t-3}) P_{t-2}/[P_{t-1} (1 + n)]\} (1 – d_h) / (1 + n) )$ to the private sector – instead of $g \{f(k_{t-1}) – f(k_{t-2}) P_{t-1}/[P_t (1 + n)]\}$, deducted in (7.3) -, with government taxes, i.e., (7.5), $T_t$ becoming $rr [f(k_{t-1}) – f(k_{t-2}) (1 – d_r) / (1 + n)] – \{f(k_{t-1}) – f(k_{t-2}) P_{t-1}/[P_t (1 + n)]\} = rr [M_t / P_t – (M_{t-1} / P_{t-1}) (1 – d_r) / (1 + n)] – dM_t / P_t$. Conclusions would not alter significantly, even though the efficient subsidies and tax rates would slightly.

The aggregate private sector problem would not alter if we assumed that at time t factor owners – consumers, expenditure makers - receive lagged income, $P_t L_t f(k_{t-1}) - dM_t /P_t = W_t L_t + R_t K_{t-1} - dM_t /P_t = W_{t-1} L_{t-1} + R_{t-1} K_{t-2}$ – then, only $(g – 1) dM_t /P_t$ is left to bear from other leakages or seigniorage rights.

A different seigniorage rights assignment when all money issuances are made against private borrowing would suggest that the central authority in real terms is left with real taxes to collect:

$$T''_t = rr [f(k_{t-1}) – f(k_{t-2}) (1 – d_r) / (1 + n)] – \{M_t / P_t – M_{t-1} / [P_t (1 + n)] + (R_t /P_t) M_{t-1} / P_{t-1} / (1 + n)\} + [M_t / P_t – M_{t-1}/P_{t-1} (1 – d) / (1 + n)] =$$
$$= rr [f(k_{t-1}) – f(k_{t-2}) (1 – d_r) / (1 + n)] – \{f(k_{t-1}) – f(k_{t-2}) [P_{t-1} / P_t - f'(k_{t-1})] / (1 + n)]\} + [M_t / P_t – M_{t-1}/P_{t-1} (1 – d) / (1 + n)] \quad (7.13)$$

Firms still pay factors at marginal product and $R_t /P_t = f'(k_{t-1})$. Profits from money transactions plus income received from capital holdings are channeled back to the private system – the term in curly brackets. The authority keeps titles or credit outstanding over the private sector of real value $M_t / P_t$ – what the private sector thinks $M_t$ is worth - at the end of transaction time t; as it had $M_{t-1}/P_{t-1} (1 – d) / (1 + n)$, the last term in squared brackets is collected (or deducted from what is returned…). Then the private sector – who works, at time t, with capital $k_{t-1}$ but only owns $k_{t-1} – D_{t-1} /$

[38] This would be consistent with demand deposits – as noted for old NOW accounts… - paying interest.

$(1 + n)$ where $D_t = M_t / P_t$ and is seen exogenous - state equation becomes:

s.t: $(1 + n) \ [k_t – D_t / (1 + n)] = (1 – d) [k_{t-1} - D_{t-1} / (1 + n)] + f(k_{t-1}) - c_t – (g + 1 – h) \{f(k_{t-1}) – f(k_{t-2}) P_{t-1}/[P_t (1 + n)]\} + (1 – h) \{f(k_{t-2}) – f(k_{t-3}) P_{t-2}/[P_{t-1} (1 + n)]\}(1 – d_h) / (1 + n) - D_{t-1} f'(k_{t-1}) / (1 + n)]\} – T''_t$ (7.14)

(To have just money covered by real asset value is another alternative. Then: $T''_t = rr [f(k_{t-1}) – f(k_{t-2}) (1 – d_r) / (1 + n)] – \{M_t / P_t – M_{t-1} / [P_t (1 + n)] + (R_t /P_t) M_{t-1} / P_{t-1} / (1 + n)\} + (1 – rr) [M_t / P_t – M_{t-1}/P_{t-1} (1 – d) / (1 + n)]$. $f'(k_{t-1})$ is multiplied by $(1 – rr)$ in (7.14) and $D_t = M_t / P_t (1 – rr)$. However, if money creation is processed through loans, the other alternative also appears reasonable.)

The equilibrium solution would be more complex that the previous one – involving second derivatives of the average product function. Still, because by selling the private sector may in fact impose some constraints on the central authority – with a partial "do it yourself", or discouragement argument – some scope for bargaining between the two extremes may occur.

Finally, we should stress the fact that if individuals realize the government/central bank budget constraint, (7.5), and internalize it, we should replace it in (7.4) before individuals' optimization. FOC then allow for the first-best solution, provided the government just follows the a monetary policy oriented towards exhausting inventories. That recognition may, in practice, be blurred or distorted due to the existence of different sides of economic activity - that were collapsed in our simple world - and uncertain incidence of taxes…

Notice also that without, or with instantaneous, money a slightly different – we can say, fully "leaded" – pricing system than usual would satisfy an efficient solution of a "production-in-advance" economy. Say that expenditure leads production according to:

$$c_t + i_t + dy_t \ = \ f(k_{t-1}) \ = \ y_t \qquad (7.15)$$

At time t, what is exchanged is

$$c_t + i_t \ = \ y_{t-1} / (1 + n) \ = \ f(k_{t-2}) / (1 + n) \qquad (7.16)$$

With CRS, a reasonable pricing system of factors used in the production of ($L_{t-1}$ $y_{t-1}$), which is exchange at time t, will obey:

$$P_t (c_t + i_t) (1 + n) = P_t y_{t-1} = P_t f(k_{t-2}) = (W_t L_{t-1} + R_t K_{t-2}) / L_{t-1} \qquad (7.17)$$

As the sale of the change in product is postponed, what effectively is paid today is yesterday's production factor use. Of course, one might argue that money requirements will (should…) respond to lagged production, i.e., the current transactions money demand is replaced by:

$$M_t = P_t y_{t-1} / (1 + n) = P_t f(k_{t-2}) / (1 + n) \qquad (7.18)$$

Then, under (7.17), i.e., without CIA restrictions (or with immediate re-insertion) or official reserves, we recover money neutrality.

Note that he definitions of $c_t$ – argument of the utility function - and $i_t$ are left as per capita values of population existing at time t, $L_t$ – therefore assuming redistribution by the current population at time t.

## 7.2. Infinitely Lived Firms and Long-Term Contracts: Q-Theories of Installed Capital and Old Labor Contracts

As $dY = F_K(K, L)\, dK + F_L(K, L)\, dL$, an obvious analogy with Tobin's (1969) q-theory of investment in the presence of adjustment costs can be made and an extension proposed [39] Let $M_t = L_t M_t$ and $M_t = P_t F(K_{t-1}, L_t)$ be replaced and allow wages to be measured in terms of real output units (new investment price is, thus, 1). Firms maximize the present value of accumulated long-term profits; they purchase new capital – investment – and pay wages being – as collateral (capital) holders - responsible for initial money issuances. Assume that rr multiplies $dY_t$ – i.e., neither reserves while such, nor the time to produce term depreciate, so that $dY_t = F_K(K_{t-1}, L_t)\, dK_{t-1} + F_L(K_{t-1}, L_t)\, dL_t$ [40]. Then, we could state a productive system problem as:

[39] We follow Barro & Sala-i-Martin (1995), p. 119-127.
[40] With exogenous technical progress, an "autonomous" term would be added.

$$\underset{I_t, dL_t, dM_t, K_t, L_t, M_t}{Max} \quad \sum_{t=1}^{\infty} \prod_{s=1}^{t} \left( \frac{1}{1+r_t} \right) \{F(K_{t-1}, L_t) \ - W_t L_t - I_t - g$$

$F(K_{t-1}, L_t)\ dM_t/M_t - rr\ [F_K(K_{t-1}, L_t)\ (I_{t-1} - d\ K_{t-2}) + F_L(K_{t-1}, L_t)\ dL_t]\}$ (7.19)

$L_t = L_{t-1} + dL_t$

$M_t = M_{t-1} + dM_t$

$\ldots \geq 0, M_t \geq 0, - F(K_{t-2}, L_{t-1}) \leq F(K_{t-1}, L_t)\ dM_t/M_t \leq F(K_{t-1}, L_t)$

Given $K_{-1}$, $K_0$, $L_0$ and $M_0$

The lagrangean form becomes:

$$\underset{I_t, dL_t, dM_t, K_t, L_t, M_t, q_{1t}, q_{2t}, q_{3t}}{Max} \quad \sum_{t=1}^{\infty} \prod_{s=1}^{t} \left( \frac{1}{1+r_s} \right) \{F(K_{t-1}, L_t) \ - W_t L_t - I_t -$$

$g\ F(K_{t-1}, L_t)\ dM_t/M_t - rr\ [F_K(K_{t-1}, L_t)\ (I_{t-1} - d\ K_{t-2}) + F_L(K_{t-1}, L_t)\ dL_t]\}$

$$+ \sum_{t=1}^{\infty} \nu_{1t}\ [- K_t + (1-d)\ K_{t-1} + I_t] + \sum_{t=1}^{\infty} \nu_{2t}\ (- L_t + L_{t-1} + dL_t) + \sum_{t=1}^{\infty} \nu_{3t}\ (- M_t + M_{t-1} + dM_t) \qquad (7.20)$$

$c_t \geq 0$, $M_t \geq 0$, $- F(K_{t-2}, L_{t-1}) \leq (M_t - M_{t-1}) / P_t = dM_t / P_t \leq F(K_{t-1}, L_t)$; given $K_{-1}$, $K_0$, $L_0$ and $M_0$

FOC imply:

$$\frac{\partial L}{\partial I_t} = \prod_{s=1}^{t} \left( \frac{1}{1+r_s} \right) [- 1 - rr \frac{1}{1+r_{t+1}} F_K(K_{t-1}, L_t)] + \nu_{1t} = 0 \qquad (7.21)$$

$$\frac{\partial L}{\partial dL_t} = - \prod_{s=1}^{t} \left( \frac{1}{1+r_s} \right) rr\ F_L(K_{t-1}, L_t) + \nu_{2t} = 0 \qquad (7.22)$$

$$\frac{\partial L}{\partial dM_t} = - \prod_{s=1}^{t} \left( \frac{1}{1+r_s} \right) g\ F(K_{t-1}, L_t) / M_t + \nu_{3t} = - \prod_{s=1}^{t} \left( \frac{1}{1+r_s} \right) g / P_t + \nu_{3t} = 0 \qquad (7.23)$$

$$\frac{\partial L}{\partial K_t} = -\nu_{1t} + (1-d)\,\nu_{1t+1} + \prod_{s=1}^{t+1}\left(\frac{1}{1+r_s}\right)\{F_K(K_t, L_{t+1})\,(1 - g\, dM_{t+1}/M_{t+1}) - rr\,[F_{KK}(K_t, L_{t+1})\,(I_t - d\,K_{t-1}) - d\,\frac{1}{1+r_{t+2}}\,F_K(K_{t+1}, L_{t+2}) + F_{LK}(K_t, L_{t+1})\,dL_{t+1}]\} = 0 \quad (7.24)$$

$$\frac{\partial L}{\partial L_t} = -\nu_{2t} + \nu_{2t+1} + \prod_{s=1}^{t}\left(\frac{1}{1+r_s}\right)\{F_L(K_{t-1}, L_t)\,(1 - g\, dM_t/M_t) - W_t - rr\,[F_{KL}(K_{t-1}, L_t)\,(I_{t-1} - d\,K_{t-2}) + F_{LL}(K_{t-1}, L_t)\,dL_t]\} = 0 \quad (7.25)$$

$$\frac{\partial L}{\partial M_t} = -\nu_{3t} + \nu_{3t+1} - \prod_{s=1}^{t}\left(\frac{1}{1+r_s}\right) g\,F(K_{t-1}, L_t)\,dM_t/M_t^2 = 0 \quad (7.26)$$

Let $q_{jt} \prod_{s=1}^{t}\left(\frac{1}{1+r_s}\right) = \nu_{jt}$, the present value of the shadow price of state variable j, j = 1,2,3. Then:

$$q_{1t} = 1 + rr\,\frac{1}{1+r_{t+1}}\,F_K(K_{t-1}, L_t) \quad (7.27)$$

$$q_{2t} = rr\,F_L(K_{t-1}, L_t) \quad (7.28)$$

$$q_{3t} = g\,F(K_{t-1}, L_t) / M_t = g / P_t \quad (7.29)$$

$$q_{1t}\,(1 + r_{t+1}) - (1-d)\,q_{1t+1} = F_K(K_t, L_{t+1})\,(1 - g\, dM_{t+1}/M_{t+1}) - rr\,[F_{KK}(K_t, L_{t+1})\,(I_t - d\,K_{t-1}) - d\,\frac{1}{1+r_{t+2}}\,F_K(K_{t+1}, L_{t+2}) + F_{LK}(K_t, L_{t+1})\,dL_{t+1}] \quad (7.30)$$

$$(1 + r_{t+1})\,q_{2t} - q_{2t+1} = (1 + r_{t+1})\,\{F_L(K_{t-1}, L_t)\,(1 - g\, dM_t/M_t) - W_t - rr\,[F_{KL}(K_{t-1}, L_t)\,(I_{t-1} - d\,K_{t-2}) + F_{LL}(K_{t-1}, L_t)\,dL_t]\} \quad (7.31)$$

$$(1 + r_{t+1})\,q_{3t} - q_{3t+1} = (1 + r_{t+1})\,g\,F(K_{t-1}, L_t)\,dM_t/M_t^2 = (1 + r_{t+1})\,g\,(dM_t/M_t) / P_t \quad (7.32)$$

The first establishes the price of installed capital owned by the firm at time (also referred at time t). The second, the price of a unitary labor contract held by the firm at time t (referred at time t) – what she will ask to another firm (over the mere release of the wage) just to grant a unit of its labor. The third, of the money balances the firm owns.

The last three equations imply:

$$r_{t+1} = (q_{1t+1} - q_{1t}) / q_{1t} - d\, q_{1t+1} / q_{1t} + \{ F_K(K_t, L_{t+1}) (1 - g\, dM_{t+1}/M_{t+1}) - rr\, [F_{KK}(K_t, L_{t+1}) (I_t - d\, K_{t-1}) - d \frac{1}{1+r_{t+2}} F_K(K_{t+1}, L_{t+2}) + F_{LK}(K_t, L_{t+1})\, dL_{t+1}]\} / q_{1t} \quad (7.33)$$

$$W_t = q_{2t+1}/(1 + r_{t+1}) - q_{2t} + \{F_L(K_{t-1}, L_t) (1 - g\, dM_t/M_t) - rr\, [F_{KL}(K_{t-1}, L_t) (I_{t-1} - d\, K_{t-2}) + F_{LL}(K_{t-1}, L_t)\, dL_t]\} \quad (7.34)$$

And

$$P_t = g\, (dM_t/M_t) / [q_{3t+1}/(1 + r_{t+1}) - q_{3t}] = (dM_t/M_t) / [(1 / P_{t+1}) / (1 + r_{t+1}) - 1 / P_t] \quad (7.35)$$

Or

$$(P_t / P_{t+1}) / (1 + r_{t+1}) = (dM_t/M_t) + 1 \quad (7.36)$$

or

$$q_{3t+1}/q_{3t} = P_t / P_{t+1} = (1 + r_{t+1}) [(dM_t/M_t) + 1] \quad (7.37)$$

The first establishes the well-known relation between the interest rate: equal to the adjusted marginal product of capital plus the expected appreciation minus depreciation of pre-existing real assets. The wage rate contains the adjusted marginal product of labor and the appreciation of pre-existing labor contracts. The last equation - – note that the term that generates it disappears if cash purchases are passed on to consumers... - can be read as $[(dM_t/M_t) + 1] = 1 / [(1 + r_{t+1}) P_{t+1} / P_t]$: the aggregate money stock increases at a rate symmetric to the (level of the) nominal interest rate. If aggregate money supply is kept fixed, the Friedman rule is reproduced – but then, in a steady-state, the real interest rate would be equal to the population growth rate (plus the growth rate of output per capita). However, with null $M_t$, condition (7.37) disappears, transformed in an inequality. And we should add $\sum_{t=1}^{\infty}$ $\nu_{4t}$ (- $Z_t + Z_{t-1}$ (1 – $d_h$) + g $F(K_{t-1}, L_t)$ $dM_t/M_t$) to the problem to generate an efficient solution…

Yet, one can argue that firms are indeed general-price takers. So the aggregate maximand should be set as [41]:

$$\underset{I_t, dL_t, dM_t, K_t, L_t, M_t}{Max} \sum_{t=1}^{\infty} \prod_{s=1}^{t} \left( \frac{1}{1+r_t} \right) \{F(K_{t-1}, L_t) - W_t L_t - I_t - g\, dM_t/P_t - rr\, [F_K(K_{t-1}, L_t)(I_{t-1} - d\, K_{t-2}) + F_L(K_{t-1}, L_t)\, dL_t]\} \tag{7.38}$$

with $P_t$ exogenously (ex-post) equal to $F(K_{t-1}, L_t) / M_t$. Then

$$q_{1t} = 1 + rr \frac{1}{1+r_{t+1}} F_K(K_t, L_{t+1}) \tag{7.39}$$

$$q_{2t} = rr\, F_L(K_{t-1}, L_t) \tag{7.40}$$

$$q_{3t} = g / P_t \tag{7.41}$$

$$q_{1t}(1 + r_{t+1}) - (1 - d)\, q_{1t+1} = F_K(K_t, L_{t+1}) - rr\, [F_{KK}(K_t, L_{t+1})(I_t - d\, K_{t-1}) - \frac{1}{1+r_{t+2}} d\, F_K(K_{t+1}, L_{t+2}) + F_{LK}(K_t, L_{t+1})\, dL_{t+1}] \tag{7.42}$$

$$(1 + r_{t+1})\, q_{2t} - q_{2t+1} = (1 + r_{t+1})\, \{F_L(K_{t-1}, L_t) - W_t - rr\, [F_{KL}(K_{t-1}, L_t)(I_{t-1} - d\, K_{t-2}) + F_{LL}(K_{t-1}, L_t)\, dL_t]\} \tag{7.43}$$

$$(1 + r_{t+1})\, q_{3t} - q_{3t+1} = 0 \tag{7.44}$$

The last three equations imply:

$$r_{t+1} = (q_{1t+1} - q_{1t}) / q_{1t} - d\, q_{1t+1} / q_{1t} + \{ F_K(K_t, L_{t+1}) - rr\, [F_{KK}(K_t, L_{t+1})(I_t - d\, K_{t-1}) - d \frac{1}{1+r_{t+2}} F_K(K_{t+1}, L_{t+2}) + F_{LK}(K_t, L_{t+1})\, dL_{t+1}]\} / q_{1t} \tag{7.45}$$

[41] One could argue that official reserves are set in nominal terms and therefore we should distinguish rr' ($M_t/P_t$ – $M_{t-1}/P_{t-1}$) + g' $dY_t$. That is irrelevant for the argument we pursue here in regarding the term multiplying rr, which deals with the last effect only – costs associated with $dY_t$.

$W_t = q_{2t+1}/(1 + r_{t+1}) - q_{2t} + \{F_L(K_{t-1}, L_t) - rr\, [F_{KL}(K_{t-1}, L_t)(I_{t-1} - d\, K_{t-2}) + F_{LL}(K_{t-1}, L_t)\, dL_t]\}$ (7.46)

and, with (7.40),

$$q_{3t+1}/q_{3t} = 1 + r_{t+1} = P_t / P_{t+1} \quad (7.47)$$

The last equation establishes that the (real, once the maximand is in real output units) shadow-prices of *nominal* money balances rise – and $P_t = g / q_{3t}$ decreases, (evolving at rate $1 / (1 + r_{t+1}) - 1$) - at the real interest rate – the present value shadow prices (referred to time 0), $v_{3t}$, will be fixed. The price of money, $1 / P_t$, increases at rate $r_t$. The rule differs from (7.35) and one recognizes the Friedman rule: a zero nominal rate of interest.

At time t the value of the firm gives to his owner profits $p_t$, the general term of the objective functions of the firm; the real value of the firm at time 0 for his owner in terms of good y is $A'_0 = \sum_{t=1}^{\infty} \prod_{s=1}^{t} \left( \frac{1}{1+r_t} \right) p_t = \frac{1}{1+r_1} p_1 + \sum_{t=2}^{\infty} \prod_{s=1}^{t} \left( \frac{1}{1+r_t} \right) p_t = \frac{1}{1+r_1} p_1 + \frac{1}{1+r_1} A'_1$. Let $A_t = A'_t / L_t$; the consumer has $A_{t-1}$ in period t and decides what to keep, $A_t$, accounting for future additional family members, i.e.:

$$\underset{c_t, A_t}{Max} \sum_{t=1}^{\infty} \rho^t\, U(c_t) \quad (7.48)$$

s.t: $w_1 + A_0\,(1 + r_1) = c_1 + A_1\,(1 + n)$ …
$w_t + A_{t-1}\,(1 + r_t) = c_t + A_t\,(1 + n)$ (7.49)

Each period, the household starts with $A_{t-1}$ in real terms, units of an asset that yields real interest $r_t$ in period t, and ends with $A_t$ $(1 + n)$ within the budget constraint, implying he buys consumption and receives wages from labor.

$$\underset{c_t, A_t, \eta_t}{Max} \sum_{t=1}^{\infty} \rho^t\, U(c_t) + \sum_{t=1}^{\infty} \eta_t\, [w_t + A_{t-1}\,(1 + r_t) - c_t - A_t\,(1 + n)]$$

(7.50)

$$\frac{\partial W}{\partial c_t} = \rho^t U_c(c_t) - \eta_t = 0 \quad \text{implying } \eta_{t+1} / \eta_t = \rho\, U_c(c_{t+1}) / U_c(c_t) \qquad (7.51)$$

$$\frac{\partial W}{\partial A_t} = - \eta_t (1 + n) + (1 + r_t)\, \eta_{t+1} = 0 \qquad (7.52)$$

Then $\rho\, U_c(c_{t+1}) / U_c(c_t) = \dfrac{1+n}{1+r_t}$; if future generations equally valued, $\rho$ is replaced by $\rho = \rho'\,(1 + n)$ and $1 + r_t = [U_c(c_t) / U_c(c_{t+1})] / \rho'$. In a steady-state, it will equal the individuals' felicity discount rate.

# 8. A BIU Growth Model

The utility function must embed two types of bequests – $k_t$ and $M_{t+1}/P_{t+1}$, the real bequests (capital and cash) that are to be passed on to the next "generation" or period after period. As $M_{t+1}/P_{t+1} = f(k_t)$, a function of $k_t$ only, preferences towards bequests can be summarized by the general inclusion of $k_t$ as an argument of the periodic utility function. People will also have preferences towards an adequate price increase: we assume they take the form of preferences for a ratio of "inheritances" to legacies of nominal balances.

We insert money in the same fashion but in a sequence of problems chain. The problem can be formalized as one of efficient point-wise decisions:

$$\underset{c_t, k_t, M_t}{Max} \quad U(c_t, k_t, M_{t-1}/M_t) \tag{8.1}$$

s.t: $(1 + n)\, k_{t+j} = (1 - d)\, k_{t+j-1} + f(k_{t+j-1}) - c_{t+j} - (g - h)\, [M_{t+j} - M_{t+j-1} / (1 + n)]\, f(k_{t+j-1}) / M_{t+j} + (1 - rr)\, (1 - h)\, f(k_{t+j-2})\, [M_{t+j-1} - M_{t+j-2} / (1 + n)] / M_{t+j-1}\, (1 - d_h) / (1 + n) - rr\, \{f(k_{t+j-1}) - f(k_{t+j-2})\, (1 - d_r) / (1 + n)]\}, \quad j = 0, 1, 2$ (8.2)

$$U(c_{t+j}, k_{t+j}, M_{t+j-1}/M_{t+j}) \geq \overline{U}_{t+j,t}\,, \quad j = 1, 2 \tag{8.3}$$

$$\text{Given } k_{t-1}, k_{t-2}, M_{t-1}, M_{t-2}$$

At time t, $\overline{U}_{t+j,t}$ is, however, still forthcoming – $\overline{U}_{t+j,t}$ would not even have to equal $\overline{U}_{t+j,t+s}$ : the corresponding multiplier should in fact be 0. Instead, the second argument of the utility function captures its concerns in a condensed manner. Therefore, we with keep the others; in lagrangean form, the problem becomes:

$$\underset{c_t, k_t, M_t, \lambda_t}{Max} \quad L(c_t, k_t, M_t, \lambda_{t,t}, \lambda_{t,t+1}, \lambda_{t,t+2}) = U(c_t, k_t, M_{t-1}/ M_t) - \sum_{j=0}^{2} \lambda_{t,\, t+j} \{k_{t+j} (1 + n) - (1 - d) k_{t+j-1} - f(k_{t+j-1}) + c_{t+j} + (g - h) [M_{t+j} - M_{t+j-1} / (1 + n)] f(k_{t+j-1}) / M_{t+j} - (1 - h) f(k_{t+j-2}) [M_{t+j-1} - M_{t+j-2} / (1 + n)] / M_{t+j-1} (1 - d_h) / (1 + n) + rr [f(k_{t+j-1}) - f(k_{t+j-2}) (1 - d_r) / (1 + n)] \}$$

F.O.C., along with the restrictions, require:

$$\frac{\partial L}{\partial c_t} = U_c(c_t, k_t, M_{t-1}/M_t) - \lambda_{t,0} = 0$$

$$\frac{\partial L}{\partial k_t} = U_k(c_t, k_t, M_{t-1}/M_t) - \lambda_{t,0} (1 + n) - \lambda_{t,1} [- (1 - d) - f'(k_t) \{1 - rr - (g - h) [M_{t+1} - M_t / (1 + n)] /M_{t+1}\}] + \lambda_{t,2} f'(k_t) \{[(1 - d_h) / (1 + n)] (1 - h) [M_{t+1} - M_t / (1 + n)] / M_{t+1} + rr (1 - d_r) / (1 + n)\} = 0$$

$$\frac{\partial L}{\partial M_t} = U_m(c_t, k_t, M_{t-1}/M_t) / M_{t-1} - \lambda_{t,0} [(g - h) M_{t-1} f(k_{t-1}) / M_t^2] / (1 + n) +$$

$$+ \lambda_{t,1} \{(g - h) (1 / M_{t+1}) f(k_t) + (1 - h) (1 - d_h) (M_{t-1} / M_t^2) f(k_{t-1}) / (1 + n)\} / (1 + n)$$

$$- \lambda_{t,2} [(1 - h) (1 - d_h) (1 / M_{t+1}) f(k_t) / (1 + n)] / (1 + n) = 0$$

Again, intertemporal evaluations would be linked to the $\lambda_{t,j}$'s. At time t, future restrictions have no direct effect on today's utility but through $\lambda_{t,0}$, that evaluates $k_t$, and are non-binding – $\lambda_{t,1}$ and $\lambda_{t,2}$ should be zero. Then, we are left with the structure:

$$\underset{c_t, k_t, M_t}{Max} \quad U(c_t, k_t, M_{t-1}/M_t) \tag{8.4}$$

s.t: $(1 + n)\, k_t = (1 - d)\, k_{t-1} + f(k_{t-1}) - c_t - (g - h)\, [M_t - M_{t-1} / (1 + n)]\, f(k_{t-1}) / M_t + (1 - rr)\, (1 - h)\, f(k_{t-2})\, [M_{t-1} - M_{t-2} / (1 + n)] / M_{t-1}\, (1 - d_h) / (1 + n) - rr\, [f(k_{t-1}) - f(k_{t-2})\, (1 - d_r) / (1 + n)]$ (8.5)

$$c_t \geq 0,\ k_t \geq 0,\ M_t \geq 0,\ M_t/M_{t-1} \geq f(k_{t-1}) / [(1 + n)\, f(k_{t-1}) + f(k_{t-2})]$$

Given $k_{t-1}, k_{t-2}, M_{t-1}, M_{t-2}$

F.O.C. of a lagrangean form, along with the current restriction only, require:

$$\frac{\partial L}{\partial c_t} = U_c(c_t, k_t, M_{t-1}/M_t) - \lambda_t = 0$$

$$\frac{\partial L}{\partial k_t} = U_k(c_t, k_t, M_{t-1}/M_t) - \lambda_t\,(1 + n) = 0$$

$$\frac{\partial L}{\partial M_t} = U_m(c_t, k_t, M_{t-1}/M_t)/M_{t-1} - \lambda_t\,(g - h)\, M_{t-1}\, f(k_{t-1}) / M_t^2 = 0$$

where $U_m$'$(c_t, k_t, M_{t-1}/M_t)$ denotes the first-derivative of $U(c_t, k_t, M_{t-1}/M_t)$ with respect to the third argument (i.e., $M_{t-1}/M_t$). Unconstrained:

$\underset{k_t, M_t}{Max}$ $U\{(1 - d)\, k_{t-1} + f(k_{t-1}) - (1 + n)\, k_t - (g - h)\, [M_t - M_{t-1} / (1 + n)]\, f(k_{t-1}) / M_t + (1 - h)\, [(1 - d_h) / (1 + n)]\, [M_{t-1} - M_{t-2} / (1 + n)]\, f(k_{t-2}) / M_{t-1} - rr\, [f(k_{t-1}) - f(k_{t-2})\, (1 - d_r) / (1 + n)],\ k_t,\ M_{t-1}/M_t\}$ (8.6)

$$k_t \geq 0,\ M_t \geq 0,\ M_t/M_{t-1} \geq f(k_{t-1}) / [(1 + n)\, f(k_{t-1}) + f(k_{t-2})]$$

Given $k_{t-1}, k_{t-2}, M_{t-1}, M_{t-2}$

F.O.C., along with the restriction, require, for t = 1, 2, 3,…,:

$$\frac{\partial U}{\partial k_t} = -\ (1 + n)\ U_c(c_t, k_t, M_{t-1}/M_t) + U_k(c_t, k_t, M_{t-1}/M_t) = 0 \quad (8.7)$$

$$\frac{\partial U}{\partial M_t} = U_c(c_t, k_t, M_{t-1}/M_t)\ \{-\ (g - h)\ (M_{t-1} / M_t^2)\ f(k_{t-1})\ \} - U_m'[c_t, k_t, f(k_t), M_{t-1}/M_t]\ (M_{t-1}/M_t^2) = 0 \quad (8.8)$$

from the first:

$$(1 + n)\ U_c(c_t, k_t, M_{t-1}/M_t) = U_k(c_t, k_t, M_{t-1}/M_t) \quad (8.9)$$

$$U_c(c_t, k_t, M_{t-1}/M_t)\ (g - h)\ f(k_{t-1}) = -\ U_m'(c_t, k_t, M_{t-1}/M_t) \quad (8.10)$$

Obviously, (8.9), (8.10) and (8.5) suggest a steady-state with constant k* and m*. The dynamic system is directly forward-looking and much simpler than for the infinitely lived economy.

Let us consider some implications for

1) A Cobb-Douglas utility function, $U(c_t, k_t, M_{t-1}/M_t) = A\ c_t^{\alpha}\ k_t^{\beta}\ (M_t / M_{t-1})^{\gamma}$, would generate:

$c_t = \dfrac{\alpha}{\beta}\ k_t\ (1 + n)$. $c_t / k_t$ increases with α and decreases with β.

And $M_t / M_{t-1} = \dfrac{\gamma}{\alpha}\ (g - h)\ f(k_{t-1}) / c_t = \dfrac{\gamma}{\beta}\ (g - h)\ f(k_{t-1}) / [k_t\ (1 + n)]$

2) A CES utility function, where one allows for $U(c_t, k_t, \dfrac{M_{t-1}}{M_t})$

$= A\left[a_1 c_t^{\rho} + a_2 k_t^{\rho} + a_3\left(\dfrac{M_{t-1}}{M_t}\right)^{\rho}\right]^{\frac{\mu}{\rho}}$ with $a_1, a_2 > 0$, $a_3 < 0$, $a_1 + a_2 + a_3 = 1$, $\rho \le 1$, would imply: $c_t = \left[\dfrac{(1+n)a_1}{a_2}\right]^{\frac{1}{1-\rho}} k_t$ – where, as is well-known, $\frac{1}{1-\rho}$ = σ corresponds to the elasticity of substitution between the two arguments. $c_t$ / $k_t$ increases with $a_1$ and decreases with $a_2$; it increases (decreases) with σ provided $\frac{(1+n)a_1}{a_2}$ > (<) 1. $M_t / M_{t-1} = \left[\frac{a_1(g-h)\, f(k_{t-1})}{-a_3}\right]^{\frac{1}{1-\rho}} / c_t$.

3) A CES utility function, where one allows for $U(c_t, k_t, \frac{M_{t-1}}{M_t})$ $= A\left[a_1 c_t^{\rho} + a_2 k_t^{\rho}\right]^{\frac{\mu}{\rho}}\left(\frac{M_t}{M_{t-1}}\right)^{\gamma}$ with $a_1, a_2 > 0$, $a_1 + a_2 = 1$, $\rho \le 1$,

would imply: $c_t = \left[\frac{(1+n)a_1}{a_2}\right]^{\frac{1}{1-\rho}} k_t$.

With exogenous technical progress, if $U(c_t, k_t, M_{t-1}/M_t)$ is homogeneous in the three arguments, condition (8.7) allows for $c_t$ and $k_t$ to move at the same proportional rate along any optimal path, and it will also be true that along it:

$$(1+n)\, U_c\left(\frac{c_t}{A_t}, \frac{k_t}{A_t}, \frac{M_{t-1}}{M_t}\right) = U_k\left(\frac{c_t}{A_t}, \frac{k_t}{A_t}, \frac{M_{t-1}}{M_t}\right) \qquad (8.11)$$

Then the problem is stated in such a way that $\frac{k_t}{A_t} = \hat{k}_t$ and $\frac{c_t}{A_t}$ $= \hat{c}_t$ and enjoy the same properties as $k_t$ and $c_t$ in the previous model: there will be a steady state *level* $\hat{k}*$ and $\hat{c}*$ that will be stable under similar requirements as before. It involves – as it does for the intertemporal utility function, neoclassical, case - a balanced-growth path for $c_t$ and $k_t$, moving at the proportional rate x per period, at which $A_t$ grows as well.

With money growth on the production side:

$\underset{k_t, M_t}{Max}$ U{(1 – d) $k_{t-1}$ + f($k_{t-1}$, $M_t/M_{t-1}$) - (1 + n) $k_t$ – (g – h) [$M_t$ – $M_{t-1}$ / (1 + n)] f($k_{t-1}$, $M_t/M_{t-1}$) /$M_t$ + (1 – h) (1 – $d_h$) [$M_{t-1}$ – $M_{t-2}$ / (1 + n)] f($k_{t-2}$, $M_{t-1}/M_{t-2}$) /$M_{t-1}$ – rr [f($k_{t-1}$) – f($k_{t-2}$) (1 – $d_r$) / (1 + n)], $k_t$} (8.12)

$k_t \geq 0$, $M_t \geq 0$, $M_t/M_{t-1} \geq f(k_{t-1}) / [(1 + n) f(k_{t-1}) + f(k_{t-2})]$
Given $k_{t-1}$, $k_{t-2}$, $M_{t-1}$, $M_{t-2}$

F.O.C., along with the restriction, require, for t = 1, 2, 3,…,:

$$\frac{\partial U}{\partial k_t} = -(1 + n)\, U_c(c_t, k_t) + U_k(c_t, k_t) = 0 \qquad (8.13)$$

$$\frac{\partial U}{\partial M_t} = (g - h)\, U_c(c_t, k_t)\, \{-(M_{t-1} / M_t^2)\, f(k_{t-1}, M_t/M_{t-1}) + \{[M_t - M_{t-1}/(1+n)]/ M_t\}\, f_m(k_{t-1}, M_t/M_{t-1}) / M_{t-1}\} = 0 \qquad (8.14)$$

$$f(k_{t-1}, 1 + m_t) / (1 + m_t) = m_t\, f_m(k_{t-1}, 1 + m_t)$$

Finally, the infinite lag adjustment generates:

$\underset{k_t, M_t, db_t}{Max}$ U{(1 – d) $k_{t-1}$ + f($k_{t-1}$) - (1 + n) $k_t$ + $db_t$ – g [$M_t$ – $M_{t-1}$ / (1 + n)] f($k_{t-1}$) /$M_t$, $k_t$, $M_{t-1}/M_t$} (8.15)

s.t.: $db_t = db_{t-1} (1 - h')(1 - d_h) / (1 + n) + h' [M_t - M_{t-1} / (1 + n)] f(k_{t-1}) / M_t$

$c_t \geq 0$, $M_t \geq 0$, $M_t/M_{t-1} \geq f(k_{t-1}) / [(1 + n) f(k_{t-1}) + f(k_{t-2})]$
Given $k_{t-1}$, $k_{t-2}$, $M_{t-1}$, $M_{t-2}$, $db_{t-1}$

Or (because each period's problem is intertemporally unconstrained by future optimization, we can replace the constraints in the objective functions…):

$$\underset{k_t, M_t}{Max} \; U\{(1 - d)\, k_{t-1} + f(k_{t-1}) - (1 + n)\, k_t + db_{t-1}\, (1 - h')\, (1 - d_h) / (1 + n) + h' [M_t - M_{t-1} / (1 + n)]\, f(k_{t-1}) / M_t - g\, [M_t - M_{t-1} / (1 + n)]\, f(k_{t-1}) / M_t, k_t, M_{t-1}/M_t\} + \quad (8.16)$$

$$c_t \geq 0,\ M_t \geq 0,\ M_t / M_{t-1} \geq f(k_{t-1}) / [(1 + n)\, f(k_{t-1}) + f(k_{t-2})]$$

$$\text{Given } k_{t-1}, k_{t-2}, M_{t-1}, M_{t-2}, db_{t-1}$$

F.O.C. require, for t = 1, 2, 3,…,:

$$\frac{\partial U}{\partial k_t} = -(1 + n)\, U_c(c_t, k_t, M_{t-1}/M_t) + U_k(c_t, k_t, M_{t-1}/M_t) = 0 \quad (8.17)$$

$$\frac{\partial U}{\partial M_t} = -\, U_c(c_t, k_t, M_{t-1}/M_t)\, \{(g - h')\, (M_{t-1} / M_t^2)\, f(k_{t-1})\, \} - U_m'[c_t, k_t, f(k_t), M_{t-1}/M_t]\, (M_{t-1}/M_t^2) = 0 \quad (8.18)$$

from the first:

$$(1 + n)\, U_c(c_t, k_t, M_{t-1}/M_t) = U_k(c_t, k_t, M_{t-1}/M_t) \quad (8.19)$$

$$U_c(c_t, k_t, M_{t-1}/M_t)\, (g - h')\, f(k_{t-1}) = -\, U_m'(c_t, k_t, M_{t-1}/M_t) \quad (8.20)$$

# 9. Optimal Fiscal and Monetary Policies; Money (M1) vs. Cash (Currency) -in-Advance

## 9.1. Endogenous Policy Parameters: Generalities

As noted in section 1, we could admit that (money…) transfers $Tr_t = h\, dM_t$ are given to private citizens per period – or that the central bank makes direct purchases from the private sector of that amount. It still had to be the case that ($dM_t$ - $Tr_t$) would be requested loans by the private sector, to be reinserted next period – or cash deposits of the central bank in commercial banks (against interest)... (1.9) would become:

$$dB_t/Q_t = Tr_t/P_t + (dM_{t-1} - Tr_{t-1})/P_{t-1}\,(1 - d_h) \qquad (9.1)$$

with exogenous $Tr_t$ but without (or replacing the role of exogenous) h… Yet, they would alter the model once $Tr_t$ cannot be then different than 0 – or pegged to $dM_t$ - in an hypothetical steady-state...

If one relies on (1.10), and allows h to be a time-variant exogenous parameter

$$dB_t/Q_t = (1 - h_{t-1})\,(1 - d_h)\,(dB_{t-1}/Q_{t-1} - Tr_{t-1}) + h_t\, dM_t/P_t + (1 - h_t)\, Tr_t \qquad (9.2)$$

the optimal monetary and fiscal policies require:

$$h_t = d_h / [1 - \rho (1 - d_h)] \qquad (9.3)$$

and

$$dM_t/P_t + (1 - h_t) Tr_t = \rho (1 - d_h) (dB_t/Q_t - Tr_t) \qquad (9.4)$$

$$Tr_t = dB_t/Q_t \ \rho [1 - \rho (1 - d_h)] / [1 - \rho^2 (1 - d_h)] \qquad (9.5)$$

$h_t$ increases with ρ and decreases with $d_h$. Also $[Tr_t / (dB_t/Q_t)]$ increases with ρ and decreases with $d_h$.

(In the presence of exogenous technical progress and population growth, they should be slightly adapted.)

Commercial banks ask loans to the central authority in response to the public's demand. They keep reserves in proportion $h_t$. Government issues nominal transfers $Tr_t$. $Tr_t$ and $h_t$ are here the (sole) government intervention parameters.

### 9.2. High-Powered Money Supply Multipliers

Let $H_t$ denote high powered money – currency plus (commercial banks) cash reserves; $h_t$ is the required reserve ratio at time t, defined as the proportion of total deposits that must be kept in cash by the bank, or cannot be lent; out of newly created cash, $f_t$ is kept by the public – currency -, the rest, (re-)deposited, allowing further loaning. In the following period, $(1 - f_t)$ is re-deposited – and $(1 - h_t)$ of it, again loaned. Then, creation of nominal supply of money is ruled by:

$$M_t - M_{t-1} / (1 + n) = Tr_t + (1 - h_t) (dH_t - Tr_t) + (1 - h_{t-1}) (1 - f_{t-1}) [Tr_{t-1} + (1 - h_{t-1}) (dH_{t-1} - Tr_{t-1})] / (1 + n) + (1 - h_{t-1}) (1 - h_{t-2}) (1 - f_{t-1}) (1 - f_{t-2}) [Tr_{t-2} + (1 - h_{t-2}) (dH_{t-2} - Tr_{t-2})] /[(1 + n) (1 + n)] + ... =$$

$$= (1 - h_{t-1}) (1 - f_{t-1}) dM_{t-1} / (1 + n) + (1 - h_t) dH_t + h_t Tr_t \qquad (9.6)$$

$Tr_t$ denotes the part of high-powered money spent by the government in nominal transfers or direct open market operations; it deposits $(dH_t - Tr_t)$, in cash (but in return of interest…), in, or grant loans to, commercial banks… We admit that is a fixed proportion of the change in high-powered money: $Tr_t = \gamma\ dH_t$ – and $\gamma$ is an exogenously fixed parameter. If all money/cash requirements are operated through the banking system, $\gamma = 0$. Note that even if $\gamma = 1$ - – only direct transfers affect high-powered money creation - would be speedier, it may not be feasible: due to (unmodelled…) economic system practices, tight money balances may firstly be felt through loans request – which the central authority in other than the deterministic environment we stage, may not even have foreseen; on the other hand, the (politically…) allowed transfers channel may not be direct: people who get the transfers may consume immediately, but to realize investment purchases, they must do it through a firm...

We could consider that in (9.6) $f_t = P_t\ c_t / M_t$, or preferably, $f_t = [P_t\ c_t - P_{t-1}\ c_{t-1} / (1 + n)] / [M_t - M_{t-1} / (1 + n)]$. Or more familiarly, $f_t = P_t\ c_t / (P_t\ y_t)$ [42], or $f_t = [P_t\ c_t - P_{t-1}\ c_{t-1} / (1 + n)] / [P_t\ y_t - P_{t-1}\ y_{t-1} / (1 + n)]$: currency in the public's hands meets the conventional cash-in-advance (towards consumption) assumption. (In fact, if annual income – GDP, based on regression of per capita values – velocity of currency is 16.3533 – of M1, 4.06849; of M2, 1.42521 -, the coefficient of the regression, without intercept, of private consumption on currency is 11.0360 and of total consumption of 13.7251, suggesting a monthly rotation period, as wage payments usually are.) We could then combine a CIA requirement for consumption expenditures, fuelled by currency [43], with a high-powered money multiplier effect – the remaining M1, narrow money supply - for investment [44]… Instead, we assume it a fixed proportion of issued currency, answering to potential investment transaction and property exchange – required at least by natural population turnover - needs.

Then the multiplier dynamics – money creation – is represented by:

[42] For Portugal, 1953-1995, the coefficient of the regression (without intercept, which was found non-significant) of private consumption on GDP (both per capita) was 0.675456; of private and public consumption expenditures, 0.841775.

[43] Nevertheless, a currency conversion requirement does not necessarily imply Clower's delay.

[44] We focus on a transactions demand for money balances – very liquid assets. In the economy, savings and time deposits would just allow services of personal property management… Yet, it is understood that by making a currency deposit, a person would be in fact acquiring capital…

$$dM_t = (1 - h)(1 - f)\, dM_{t-1} / (1 + n) + [1 - h(1 - \gamma)]\, dH_t \quad (9.7)$$

with $dM_t = M_t - M_{t-1} / (1 + n)$ and $dH_t = H_t - H_{t-1} / (1 + n)$. It is a second-order differential version of the monetary base multiplier. In a model where $dM_t$ – or rather $M_t$ - is required at time t, with previous settlement of $M_t$' and $H_t$' of earlier periods, (9.7) provides the required $dH_t$.

On the one hand, if in a steady state, $dM_t$ grows at the same rate, m*, as real per capita money balances, as then $(dM_t / dM_{t-1})^* = 1 + m^*$, - and $dM_t$ and $dH_t$ cannot be systematically negative - (9.7) implies an implicit long-run money multiplier obeying

$$dM_t = [1 - h(1 - \gamma)]\, dH_t \; / \; \{1 - (1 - h)(1 - f) / [(1 + m^*)(1 + n)]\} \quad (9.8)$$

If $\gamma = 1$, $dM_t / dH_t = 1 / \{1 - (1 - h)(1 - f) / [(1 + m^*)(1 + n)]\}$.

On the other, (9.7) comes from an aggregate definition; it suggests – integrating – that [45]:

$$M_t = a / L_t + (1 - h)(1 - f)\, M_{t-1} / (1 + n) + [1 - h(1 - \gamma)]\, H_t \quad (9.9)$$

(Again we can confirm the long-run multiplier (9.8) if $M_t$ grows at a stable rate m*…)

Given past values of $H_0$, $M_{-1}$ and, $M_0$, if the mechanism is exogenous and stable, $M_0 = a / L_0 + (1 - h)(1 - f)\, M_{-1} / (1 + n) + [1 - h(1 - \gamma)]\, H_0$ and, therefore,

$$a / L_0 = M_0 - (1 - h)(1 - f)\, M_{-1} / (1 + n) - [1 - h(1 - \gamma)]\, H_0. \quad (9.10)$$

$a = 0$ occurs iff $L_0 M_0 = (1 - h)(1 - f)\, L_{-1} M_{-1} + [1 - h(1 - \gamma)]\, L_0 H_0$. Being that the case, (9.9) – without the first term of the

[45] For Portugal, 1948-1995 – using information from Pinheiro *et al.,* (1997) -, the intercept of the aggregate regression was found insignificant (with a p-value of 17.9% - 18.6% when per capita aggregates were used); coefficients of the regression (without intercept, 1954-1995) (9.9) using M1 – currency plus demand deposits – in per capita terms were, respectively, 1.06769 – which should be smaller than 1… - and 0.062594.

right hand-side – can replace (9.7), and we can say we have a first-order form of the multiplier.

When (9.7) is introduced in a model targeting $m_t$, or is staged in the presence of exogenous dynamics of $L_t H_t$:

1) If $a / (H_t L_t) = 0$, or tends to zero, i.e., $H_t L_t$ tends to infinity and ($H_t/M_t$ tends to a constant) $1 + m^* > 1 / (1 + n)$ so that aggregate money balances increase continually, (9.8) and (9.9) also suggest a constant long-run – steady-state - ratio (H / M):

$$(H / M)^* = \{1 - (1 - h)(1 - f) / [(1 + m^*)(1 + n)]\} / [1 - h(1 - \gamma)] \quad (9.11)$$

$(H / M)^*$ increases with h, f, $m^*$ and n; it decreases with $\gamma$.

2) If $a / (H_t L_t)$ tends to infinity – because $(H_t L_t)$ tends to zero (say, $m^* < 1 / (1 + n) - 1$) -, $(H / M)^*$ will tend to $\{1 - (1 - h)(1 - f) / [(1 + m^*)(1 + n)]\} / \{a / (L_t H_t) + [1 - h(1 - \gamma)]\}$, which will tend to zero.

3) Finally, if $(H_t L_t)$ tends to a constant different than zero, $(H_t L_t)^*$ - suggesting $m^* = 1 / (n + 1) - 1$ exactly - , $(H / M)^*$ will tend to $[1 - (1 - h)(1 - f)] / \{a / (L_t H_t)^* + [1 - h(1 - \gamma)]\}$. It then increases with $(L_t H_t)^*$.

Provided a steady state supports $m^* > 1 / (1 + n) - 1$, (9.7), the second-order differential equation of the multiplier can be used. If not, the first difference one could (if justified empirically…), (9.9) with a = 0.

An alternative process can be justified by a distributed lag adjustment similar to (9.6) but working in levels instead of changes: suppose that new issued monetary base, out of which ? is immediately added to money as currency – by transfers or direct purchases of the central bank - and (1 – h) of the remainder lent to the private sector – h of that remainder left as commercial banks reserves -, accrues to previous money balances. In period t, adjustment of previous money balances to a fixed proportion of past monetary base, such that they equal $H_{t-1}$ times the long-run multiplier (under constant aggregate supply), is already accomplished – all the accumulated effect of total $H_{t-1}$ is achieved in one period of time -, and added of new currency so that [46] [47]:

$$M_t = H_{t-1} \{[1 - h (1 - \gamma)] / [1 - (1 - h) (1 - f)]\} / (1 + n) + [1 - h (1 - \gamma)] dH_t =$$
$$= [1 - h (1 - \gamma)] (H_t + \{[(1 - h) (1 - f)] / [1 - (1 - h) (1 - f)]\} H_{t-1} / (1 + n)) \qquad (9.12)$$

The new multiplier if H and M are to grow at the same rate m* is - the same as that of the second-order difference one -, therefore:

$$(M / H)^* = [1 - h (1 - \gamma)] (1 + [(1 - h) (1 - f)] / \{[1 - (1 - h) (1 - f)] / [(1 + n) (1 + m^*)] \}) \qquad (9.13)$$

. Finally, notice that real reserve creation is considered outside the high-powered money supply process. Under full convertibility, an additional multiplier – linked to the real required reserved ratio rr - could generate high-powered money creation through the commercial banking system, stemming from official real reserve changes – deposits of which, or species itself - solicited by the government in exchange for currency… If the nominal unit is not indexed in species, seigniorage would stem from its appreciation, from currency depreciation; if it is ("break" in currency not allowed…), theoretically, there are no seigniorage rights - provided gold depositors receive interest, as any other, that the central bank would also ask for cash loans…)…

[46] For Portugal, 1954-1995 – using information from Pinheiro *et al.,* (1997) -, the intercept of the aggregate regression was found almost significant (with a p-value of 10.8% when per capita aggregates were used); coefficients of the regression (without intercept, 1954-1995) (9.12) using M1 – currency plus demand deposits – in per capita terms were, respectively, 0.386621 of $H_t$ and 1.01356 of $H_{t-1}$.

[47] For Portugal, 1954-1995 – using information from Pinheiro *et al.,* (1997) -, the intercept of the aggregate regression was found insignificant (with a p-value of 29.6% when per capita aggregates were used); coefficients of the regression (without intercept, 1954-1995) (9.12) using M2 – currency plus demand deposits – in per capita terms were, respectively, 0.833438 of $H_t$ and 3.16575 of $H_{t-1}$.

### 9.3. Money-in-Advance

#### 9.3.1. "Unit-of-Account Neutrality"

The mechanism (9.6) can reproduce money dynamics – with a required cash reserve constraint - in an economy where transactions can be operated through bank transfers of individuals' accounts balances. Suppose money transaction requirements – embedded in (1.2) - do not have to be met by currency, yet they must by money: to make a purchase of final product, $f_t$ is paid in advance in cash – which is withdrawn from the commercial bank deposits and out of them for the period; people do not have to hold the $(1 – f_t)$ proportion in cash, but must have, prove they have it – or rather, its worth – in advance; then, they just have to keep this proportion deposited – immobilized - in the bank during the period.

The model replaces (1.6) by:

$$c_t + i_t + rr_t\, H_t/P_t - rr_{t-1}\, H_{t-1}/P_{t-1}\,(1 - d_r) + (g - 1)\, dM_t/P_t + g' [f(k_{t-1}) - f(k_{t-2})\,(1 - d_y) / (1 + n)] = f(k_{t-1}) \quad (9.14)$$

g' denotes the hypothetical adjustment losses with the production system operation – now, we can no longer consider them added to the required reserve ratio (i.e., rr = rr' + g' for comparison purposes), which we now denote rr'... Still, $(g – 1)\, dM_t/P_t$ represents for g = 2 a "money-in-advance" assumption that can still be superimposed: the money creation and conversion mechanism would replace $(dB_t/Q_t – g\, dM_t/P_t)$ of the previous generalized CIA modeling by $(g – 1)\, dM_t/P_t$. We leave g" = (g – 1). $g' [f(k_{t-1}) - f(k_{t-2})\,(1 – d_y) / (1 + n)]$ is considered accruing to inventory build-up – and added to its state equation; we could have just assumed them a loss to production instead (i.e., deduct the term from the capital state equation but not add them to the inventory one.).

One could think that cash-in advance delays no longer apply to total money balances but only to newly issued currency, i.e., that $g"\, dM_t/P_t$ should eventually be replaced by $g"\, dH_t/P_t$; yet, (9.6) rules the money supply process: it may still be the case that transactions delay are (technologically) demand induced by production operating rhythm. Conversion delays could still imply an expenditure leakage working through $(1 - \gamma)\, dH_t/P_t$ only – and we could then deduct $(1 - h)\,(1 - \gamma)\,[dH_t/P_t - dH_{t-1}/P_{t-1}\,(1 - d_h) / (1 + n)]$ from the capital state equation; but could also apply to total new money, $M_t$, creation – or yet to $[dM_t - \gamma\ dH_t]/P_t$... (Seigniorage would only need to be modeled to derive an equilibrium, not the efficient allocation. It only stems from high-powered money – provided commercial bank deposits pay interest…) To simplify matters, we ignore then.

The multiplier is introduced in a second-order differenced form [48]. The planner's problem becomes:

$$\underset{c_t,k_t,dH_t,H_t,dM_t,M_t,P_t,z_t}{Max} \sum_{t=1}^{\infty} \rho^t\ U(c_t)$$

s.t: $(1 + n)\ k_t = (1 - d)\ k_{t-1} + f(k_{t-1}) - c_t - g''\ dM_t/P_t - g'\ [f(k_{t-1}) - f(k_{t-2})\ (1 - d_y) / (1 + n)] - rr'\ [H_t/P_t - H_{t-1}/P_{t-1}\ (1 - d_r) / (1 + n)]$ (9.15)

$dM_t = (1 - h)\ (1 - f)\ dM_{t-1} / (1 + n) + [1 - h\ (1 - \gamma)]\ dH_t$ (9.16)

$H_t = H_{t-1} / (1 + n) + dH_t$ (9.17)

$M_t = M_{t-1} / (1 + n) + dM_t$ (9.18)

$M_t = P_t\ f(k_{t-1})$ (9.19)

$z_t = z_{t-1}\ (1 - d_h) / (1 + n) + g''\ dM_t/ P_t + g'\ [f(k_{t-1}) - f(k_{t-2})\ (1 - d_y) / (1 + n)]$ (9.20)

$c_t \geq 0,\ k_t \geq 0,\ M_t \geq 0,\ H_t \geq 0,\ z_t \geq 0,\quad - f(k_{t-2}) / (1 + n) \leq dM_t / P_t \leq f(k_{t-1})$

Given $k_{-1}, k_0, H_0, M_0, dM_0, P_0, z_0$

The Hamiltonian analog would be linear in $dM_t$ but also in $dH_t$. Then we do not expect in compact forms of the problem interior solutions for either $M_t$ or $H_t$. And if $H_t$ is just dictated by $M_t$ and not a corner (i.e., different from 0), we must (or rather, may: we are applying rules of a first-order Hamiltonian…) be in the presence of a singular solution for H.

It can be further simplified to:

[48] For Portugal, 1949-1995 – using information from Pinheiro *et al.,* (1997) -, the coefficients of the regression (without intercept) (9.16) using M1 – currency plus demand deposits – as the money aggregate were, respectively, 0.940591 – approaching (1 – h) (1 – f) for an annual revolving period - and 0.00788682 (if one uses the interpretation of appendix B, and regress accordingly, $[(1 - h)\ (1 - f)]^j$ equals 0.958267; yet the second term is negative even if insignificant). The long run ratio (using per capita aggregates) (M1 / H) for 1953-1995 was 1.32133.

$$\underset{k_t,H_t,M_t}{Max} \sum_{t=1}^{\infty} \rho^t \; U\{(1-d)\, k_{t-1} + f(k_{t-1}) - (1+n)\, k_t - g'' f(k_{t-1}) [M_t - M_{t-1} / (1+n)]/M_t - g' [f(k_{t-1}) - f(k_{t-2}) (1-d_y) / (1+n)] - rr' [f(k_{t-1}) H_t / M_t - f(k_{t-2}) (H_{t-1}/ M_{t-1}) (1-d_r) / (1+n)]\} \qquad (9.21)$$

$$\text{s.t.:}\; M_t = M_{t-1} / (1+n) + [(1-h)(1-f)/(1+n)] [M_{t-1} - M_{t-2} / (1+n)] + [1-h(1-\gamma)] [H_t - H_{t-1} / (1+n)] \qquad (9.22)$$

$$z_t = z_{t-1} (1-d_h) / (1+n) + g'' dM_t/ P_t + g' [f(k_{t-1}) - f(k_{t-2}) (1-d_y) / (1+n)] \qquad (9.23)$$

$$c_t \geq 0,\ H_t \geq 0,\ M_t \geq 0,\ z_t \geq 0; \text{ given } k_{-1}, k_0, H_0, M_{-1}, M_0, z_0$$

Looking at the structure of the objective function, one immediately concludes that $H_t / M_t$ is consumption detracting.

$$\underset{k_t,H_t,M_t,\mu_t}{Max} \; L = \sum_{t=1}^{\infty} \rho^t \; U\{(1-d)\, k_{t-1} + f(k_{t-1}) - (1+n)\, k_t - g'' f(k_{t-1}) [M_t - M_{t-1} / (1+n)]/M_t - g' [f(k_{t-1}) - f(k_{t-2}) (1-d_y) / (1+n)] - rr' [f(k_{t-1}) H_t / M_t - f(k_{t-2}) (H_{t-1}/ M_{t-1}) (1-d_r) / (1+n)]\} + \sum_{t=1}^{\infty} \mu_t (M_t - \{M_{t-1} / (1+n) + [(1-h)(1-f) / (1+n)] [M_{t-1} - M_{t-2} / (1+n)] + [1-h(1-\gamma)] [H_t - H_{t-1} / (1+n)]\} ) + \sum_{t=1}^{\infty} \eta_t \{- z_t + z_{t-1} (1-d_h) / (1+n) + g'' f(k_{t-1}) [M_t - M_{t-1} / (1+n)] / M_t + g' [f(k_{t-1}) - f(k_{t-2}) (1-d_y) / (1+n)] \} \qquad (9.24)$$

FOC generate:

$$\frac{\partial W}{\partial k_t} = \rho^t ( - (1+n) U_c(c_t) + \rho\, U_c(c_{t+1}) [(1-d)+ f'(k_t) \{1 - g' - g'' [M_{t+1} - M_t / (1+n)] /M_{t+1} - rr' H_{t+1} / M_{t+1} \}] + \rho^2 U_c(c_{t+2}) f'(k_t) [ rr' (1-d_r) / (1+n) H_{t+1}/ M_{t+1} + g' (1-d_y) / (1+n)] ) + \eta_{t+1} f'(k_t) \{g'' [M_t - M_{t-1} / (1+n)] / M_t + g'\} - \eta_{t+2}\, g' f'(k_t) (1-d_y) / (1+n) = 0 \qquad (9.25)$$

$$\frac{\partial W}{\partial H_t} = \rho^t \{ - rr' U_c(c_t) (1 / M_t) f(k_{t-1}) + \rho U_c(c_{t+1}) rr' f(k_{t-1}) (1 / M_t) (1 - d_r) / (1 + n)\} - \mu_t [1 - h (1 - \gamma)] + \mu_{t+1} \{[1 - h (1 - \gamma)] / (1 + n)\} = 0 \quad (9.26)$$

$$\frac{\partial W}{\partial M_t} = \rho^t \{ [- g'' M_{t-1} / (1 + n) + rr' H_t] U_c(c_t) (1 / M_t^2) f(k_{t-1}) + \rho U_c(c_{t+1}) \{ [g'' / (1 + n)] f(k_t) / M_{t+1} - rr'' f(k_{t-1}) (H_t / M_t^2) (1 - d_r) / (1 + n)\} + \mu_t - \mu_{t+1} [(1 - h) (1 - f)] / (1 + n) + \mu_{t+2} [(1 - h) (1 - f)] / (1 + n)^2 + \eta_t \{ g'' f(k_{t-1}) (M_{t-1} / M_t^2) / (1 + n) - \eta_{t+1} \{ g'' f(k_t) (1 / M_{t+1}) / (1 + n) = 0 \quad (9.27)$$

$$\frac{\partial W}{\partial z_t} = - \eta_t + \eta_{t+1} (1 - d_h) / (1 + n) = 0 \quad (9.28)$$

For a steady-state with a constant k*, it is suggested that $dM_t/P_t$ should be constant. Then $dM_t$ would grow at the same rate as the general price level. Being f constant, that would imply that $dM_t$ would also grow at the same rate as real per capita money balances, or be zero. Then the implicit long-run money multiplier obeys (9.8) and – if a = 0 or similar – (9.11) will hold.

Reasoning as in section 2, one would be led to the conclusion that a pursuit of null inventories could be an optimal policy; yet, due to the structure of the official reserve deduction, dependent on $H_t$ and $M_t$ and lags, there may be paths where – at least temporarily - trade-offs between that term and inventory build-up may dictate interior solutions for the latter. Nevertheless, monetary growth is expected to harm welfare prospects doubly in the long-run – through inventories, but also through H/M which, in a steady-state, increases with m; and any attempt to drive m down will always be bounded from below by the non-negative inventory requirement…

Under null inventories – from (9.25) - capital, $k_t$, and consumption, $c_t$, follow a path consistent with:

$f'(k_t) = [(1 + n)\, U_c(c_t) - \rho\, U_c(c_{t+1})\, (1 - d)] / [\rho\, U_c(c_{t+1})\, (1 - rr'\, H_{t+1} / M_{t+1}) + \rho^2\, U_c(c_{t+2})\, rr'\, H_{t+1} / M_{t+1}\, (1 - d_r)/(1 + n)]$ (9.29)

and

$c_t = (1 - d)\, k_{t-1} + f(k_{t-1}) - (1 + n)\, k_t - rr'\, [f(k_{t-1})\, H_t/M_t - f(k_{t-2})\, H_{t-1}/M_{t-1}\, (1 - d_r) / (1 + n)]$ (9.30)

The zero inventory policy can be accomplished if the central authority targets $M_1$ such that

$0 = z_1 = z_0\, (1 - d_h) / (1 + n) + g''\, [M_1 - M_0 / (1 + n)]\, f(k_0) / M_1 + g'[f(k_0) - f(k_{-1})\, (1 - d_h) / (1 + n)]$

fixing thus $H_1$ such that

$[M_1 - M_0 / (1 + n)] = (1 - h)\, (1 - f)\, dM_0 / (1 + n) + [1 - h\, (1 - \gamma)]\, [H_1 - H_0 / (1 + n)]$

For $t = 2,3...\ z_t = 0$ the authority sets $M_t$ such that:

$g''\, f(k_{t-1})\, [M_t - M_{t-1} / (1 + n)] / M_t = - g'\, [f(k_{t-1}) - f(k_{t-2})\, (1 - d_h) / (1 + n)]$ (9.31)

or

$M_t = [M_{t-1} / (1 + n)] / (1 + (g'/g'')\, \{1 - [f(k_{t-2})/ f(k_{t-1})]\, (1 - d_h) / (1 + n)\})$ (9.32)

systematically requiring $H_t$ derived from:

$[H_t - H_{t-1} / (1 + n)] = \{[M_t - M_{t-1} / (1 + n)] - (1 - h)\, (1 - f)\, [M_{t-1} - M_{t-2} / (1 + n)] / (1 + n)\} / [1 - h\, (1 - \gamma)] = (g'/g'')\, ( - M_t\, \{1 - [f(k_{t-2})/ f(k_{t-1})]\, (1 - d_h) / (1 + n)\} + [(1 - h)\, (1 - f) / (1 + n)]\, M_{t-1}\, \{1 - [f(k_{t-3})/ f(k_{t-2})]\, (1 - d_h) / (1 + n)\} ) / [1 - h\, (1 - \gamma)]$ (9.33)

(9.32) suggests that for a null inventory sequence in the long-run, $M_t / M_{t-1}$ would tend to $1 / [1 + n + (g' / g'')\, (n + d_h)] < 1 / (1$

+ n); but then (9.33) would imply that we will achieve zero $dH_t$ – after which the zero inventory policy is no longer attainable. Then the inventory state equation is no longer relevant and we would apply:

$$f'(k_t) = [(1 + n)\, U_c(c_t) - \rho\, U_c(c_{t+1})\,(1 - d)] \,/\, \{ \rho\, U_c(c_{t+1})\,[1 - rr'\, H_{t+1} / M_{t+1} - g'] + \rho^2\, U_c(c_{t+2})\,[g'\,(1 - d_y) / (1 + n) + rr'\,(H_{t+1} / M_{t+1})\,(1 - d_r) / (1 + n)] \} \quad (9.34)$$

and

$$c_t = (1 - d)\, k_{t-1} + f(k_{t-1}) - (1 + n)\, k_t - rr'\, [f(k_{t-1})\, H_t/M_t - f(k_{t-2})\, H_{t-1}/M_{t-1}\,(1 - d_r) / (1 + n)] - g'\,[f(k_{t-1}) - f(k_{t-2})\,(1 - d_y) / (1 + n)] \quad (9.35)$$

In the steady-state:

$$f'(k^*) = [(1 + n) - \rho\,(1 - d)] \,/\, \{ \rho\,[1 - g' - rr'\,(H/M)^*] + \rho^2\, [rr'\,(H/M)^*\,(1 - d_r) / (1 + n) + g'\,(1 - d_y) / (1 + n)] \} \quad (9.36)$$

$$c^* = f(k^*)\, \{1 - rr'\,(H/M)^*\,[1 - (1 - d_r) / (1 + n)] - g'\,[1 - (1 - d_y) / (1 + n)]\} - (n + d)\, k^* \quad (9.37)$$

From then, we conclude that steady-state capital as per capita consumption would decrease with $(H/M)^*$, $g'$ and $d_y$.

But if H reached zero, $(H / M)^*$ may have been driven to zero (case 2 of end of previous sub-section is not a possibility…), eventually replaceable in the above…

Interesting variants or special cases of the problem present dilemmas with respect to the multiplier:

1) If $g' = 0$, we could infer $H_1$ and $M_1$ – in a policy where inventories are set to zero as soon as possible - from:

$$0 = z_1 = z_0\,(1 - d_h) / (1 + n) + g''\,[M_1 - M_0 / (1 + n)]\, f(k_0) / M_1$$

$$[M_1 - M_0 / (1 + n)] = (1 - h)\,(1 - f)\, dM_0 / (1 + n) + [1 - h\,(1 - \gamma)]\,[H_1 - H_0 / (1 + n)]$$

Then $H_2$ would obey:

$0 = [M_2 – M_1 / (1 + n)] = (1 – h) (1 – f) [M_1 – M_0 / (1 + n)] / (1 + n) + [1 – h (1 - \gamma)] [H_2 – H_1 / (1 + n)]$

(and
$0 = [M_3 – M_2 / (1 + n)] = (1 – h) (1 – f) [M_2 – M_1 / (1 + n)] / (1 + n) + [1 – h (1 - \gamma)] [H_3 – H_2 / (1 + n)] = [H_3 – H_2 / (1 + n)]$
...)

$H_t\ L_t$ would then be kept fixed at the level ($H_2\ L_2$); $M_t\ L_t$, at level $L_2\ M_2$ (then $M_2 = M_1 / (1 + n)$) – we could deduct the ratio to apply in (9.29). The steady-state would then not be independent of initial conditions – and would differ from the Ramsey result:

$$f'(k^*) = [(1 + n) / \rho - (1 – d)] / \{1 – rr' (H_t / M_t)^* [1 - \rho (1 – d_r) / (1 + n)]\} \qquad (9.38)$$

and

$$c^* = f(k^*) \{1 - rr' (H_t/M_t)^* [1 – (1 - d_r) / (1 + n)]\} – (d + n) k^* \qquad (9.39)$$

Moreover, there may be now a trade-off between change in inventories and in official reserves that the planner may be able to exploit – the Hamiltonian is no longer linear in, now, $dH_t$.

Targeting $z_t = 0$, $t = 2,3,\ldots$ and allowing $z_1$ to be positive instead would require:

$[M_1 – M_0 / (1 + n)] = (1 – h) (1 – f) dM_0 / (1 + n) + [1 – h (1 - \gamma)] [H_1 – H_0 / (1 + n)]$

$[M_2 – M_1 / (1 + n)] = (1 – h) (1 – f) [M_1 – M_0 / (1 + n)] / (1 + n) + [1 – h (1 - \gamma)] [H_2 – H_1 / (1 + n)] = 0$

$z_1 = z_0 (1 – d_h) / (1 + n) + g'' [M_1 – M_0 / (1 + n)] f(k_0) / M_1$
$0 = z_2 = z_1 (1 – d_h) / (1 + n)$

The last two equations determine $z_1$ and $M_1$. Then the first two determine $H_1$ and $H_2$. We have a similar problem…

2) Suppose a delayed re-insertion mechanism – now of all money creation - is in place. We have a term h" $(1 - d_h) / (1 + n)$ $dM_{t-1} / P_{t-1}$ added to the state equation – g' is set to zero. Then, the zero inventory driven policy can be accomplished if the central authority targets $M_1$ such that

$$0 = z_1 = z_0 (1 - d_h) / (1 + n) + g'' [M_1 - M_0 / (1 + n)] f(k_0) / M_1 - h'' (1 - d_h) / (1 + n) dM_0 / P_0$$

fixing thus $H_1$ such that

$$[M_1 - M_0 / (1 + n)] = (1 - h) (1 - f) dM_0 / (1 + n) + [1 - h (1 - \gamma)] [H_1 - H_0 / (1 + n)]$$

For t = 2,3... $z_t = 0$ the authority sets $M_t$ such that:

$$g'' [M_t - M_{t-1} / (1 + n)] f(k_{t-1}) / M_t = h'' [(1 - d_h) / (1 + n)] [M_{t-1} - M_{t-2} / (1 + n)] f(k_{t-2}) / M_{t-1}$$

or

$$[M_t - M_{t-1} / (1 + n)] / M_t = [f(k_0)/ f(k_{t-1})] \{[(h''/g'') (1 - d_h) / (1 + n)]\}^{t-1} \{[M_1 - M_0 / (1 + n)] / M_1\}$$

or

$$M_t = [M_{t-1} / (1 + n)] / (1 - [f(k_0)/ f(k_{t-1})] \{[(h''/g'') (1 - d_h) / (1 + n)]\}^{t-1} \{[M_1 - M_0 / (1 + n)] / M_1\})$$

systematically requiring $H_t$:

$$[H_t - H_{t-1} / (1 + n)] = \{[M_t - M_{t-1} / (1 + n)] - (1 - h) (1 - f) [M_{t-1} - M_{t-2} / (1 + n)] / (1 + n)\} / [1 - h (1 - \gamma)] = f(k_0) \{ [M_t / f(k_{t-1})] [(h''/g'') (1 - d_h) / (1 + n)] - [(1 - h) (1 - f) / (1 + n)] [M_{t-1} /$$

$f(k_{t-2})]\ \}\ \{[(h''/g'')\ (1 - d_h) / (1 + n)]\}^{t-2}\ \{[M_1 - M_0 / (1 + n)] / M_1\}\ / [1 - h\ (1 - \gamma)]$

or

$$H_t = f(k_0)\ \{[M_1 - M_0 / (1 + n)] / M_1\} \sum_{j=1}^{t-2} \{[M_{t-j+1} / f(k_{t-j+1-1})]\ [(h''/g'')\ (1 - d_h) / (1 + n)] - [(1 - h)\ (1 - f) / (1 + n)]\ [M_{t-j+1-1} / f(k_{t-j+1-2})]\}\ \{[(h''/g'')\ (1 - d_h) / (1 + n)]\}^{t-j+1-2} / \{[1 - h\ (1 - \gamma)]^j\ (1 + n)^{j-1}\} + H_1 / (1 + n)$$

$M_t / M_{t-1}$ will tend to $1 / (1 + n)$; we will end-up with $(H / M)^*$ given by:

$$(H / M)^* = [1 - (1 - h)\ (1 - f)] / [1 - h\ (1 - \gamma)] \qquad (9.40)$$

that can therefore be replaced in (9.34) and (9.35) (for g' = 0).

3) Finally, note that if the term $g'\ [f(k_{t-1}) - f(k_{t-2})\ (1 - d_y) / (1 + n)]$ is a pure production loss (or enhancement if negative) and does not represent storable items, it is not added to the inventories. Then, a zero-inventory target leads to $H_2$ and $M_2$ of case 1) and the economy follows the real path (9.34) and (9.35); the steady-state is given by (9.36) and (9.37) for $H_t = H_2$ and $M_t = M_2$.

Suppose, therefore, that we stage instead a first-order difference multiplier. The planner's problem is:

$$\underset{c_t,k_t,dH_t,H_t,dM_t,M_t,P_t,z_t}{Max} \sum_{t=1}^{\infty} \rho^t\ U(c_t)$$

s.t: $(1 + n)\ k_t = (1 - d)\ k_{t-1} + f(k_{t-1}) - c_t - g''\ dM_t/P_t - g'\ [f(k_{t-1}) - f(k_{t-2})\ (1 - d_y) / (1 + n)] - rr'\ [H_t/P_t - H_{t-1}/P_{t-1}\ (1 - d_r) / (1 + n)]$ (9.41)

$$M_t = (1 - h)\ (1 - f)\ M_{t-1} / (1 + n) + [1 - h\ (1 - \gamma)]\ H_t \qquad (9.42)$$

$$M_t = M_{t-1} / (1 + n) + dM_t \qquad (9.43)$$

$$M_t = P_t\ f(k_{t-1}) \qquad (9.44)$$

$$z_t = z_{t-1}\ (1 - d_h) / (1 + n) + g''\ dM_t/ P_t + g'\ [f(k_{t-1}) - f(k_{t-2})\ (1 - d_y) / (1 + n)] \qquad (9.45)$$

$c_t \geq 0$, $k_t \geq 0$, $M_t \geq 0$, $H_t \geq 0$, $z_t \geq 0$, $- f(k_{t-2}) / (1 + n) \leq dM_t / P_t \leq f(k_{t-1})$

Given $k_{-1}$, $k_0$, $H_0$, $M_0$, $dM_0$, $P_0$, $z_0$

The economy's new optimal path would be attainable with the second-order process if a = 0 – i.e., if initial stocks allow (9.10) to be zero.

(9.29) and (9.30) hold with a zero inventory policy; it can be accomplished if the central authority targets $M_1$ such that

$$0 = z_1 = z_0 (1 - d_h) / (1 + n) + g'' [M_1 - M_0 / (1 + n)] f(k_0) / M_1 + g'[f(k_0) - f(k_{-1}) (1 - d_h) / (1 + n)]$$

fixing thus $H_1$ such that

$$M_1 = (1 - h) (1 - f) M_0 / (1 + n) + [1 - h (1 - \gamma)] H_1$$

For t = 2,3... $z_t = 0$ the authority sets $M_t$ such that:

$$g'' f(k_{t-1}) [M_t - M_{t-1} / (1 + n)] / M_t = - g' [f(k_{t-1}) - f(k_{t-2}) (1 - d_h) / (1 + n)] \tag{9.46}$$

or

$$M_t = [M_{t-1} / (1 + n)] / (1 + (g'/g'') \{1 - [f(k_{t-2}) / f(k_{t-1})] (1 - d_h) / (1 + n)\}) \tag{9.47}$$

systematically requiring $H_t$ derived from:

$$\begin{aligned} H_t &= M_t - (1 - h) (1 - f) [M_{t-1} / (1 + n)] / [1 - h (1 - \gamma)] = \\ &= [M_{t-1} / (1 + n)] \{1 / (1 + (g'/g'') \{1 - [f(k_{t-2}) / f(k_{t-1})] (1 - d_h) / (1 + n)\}) - (1 - h) (1 - f) / [1 - h (1 - \gamma)] \} = \\ &= M_t \{ 1 - (1 - h) (1 - f) (1 + (g'/g'') \{1 - [f(k_{t-2}) / f(k_{t-1})] (1 - d_h) / (1 + n)\}) / [1 - h (1 - \gamma)] \} \end{aligned} \tag{9.48}$$

Then, $m^* = 1 / [1 + n + (g'/g'') (n + d_h)] - 1 < 1 / (1 + n) - 1$, and it is possible to converge to:

$$(H_t / M_t)^* = (1 - (1 – h) (1 – f) \{1 + (g'/g'') [(n + d_h) / (1 + n)]\}) / [1 – h (1 - \gamma)] \quad (9.49)$$

provided it is larger than 0. In that case, :

$$f'(k^*) = [(1 + n) - \rho (1 – d)] / \{\rho [1 – rr' (H/M)^*] + \rho^2 [rr' (H/M)^* (1 – d_r) / (1 + n) ]\} \quad (9.50)$$

As $\rho (1 – d_r) / (1 + n) < 1$, $k^*$ decreases with $(H/M)^*$ - because $f''(k) < 0$; therefore (not unexpectedly…) it decreases with f and h and it increases with $\gamma$. As $(H/M)^* < 1$ – and comparing with (3.17) for $rr = rr' + g'$ -, $k^*$ is expected to be higher with a high-powered money supply for a required (commercial) reserve ratio $h < 1$ (and $f < 1$): the mechanism saves in terms of required "real" official reserves…

Consumption could be obtained from:

$$c^* = f(k^*) \{1 - rr' (H/M)^* [1 – (1 – d_r) / (1 + n)]\} – (n + d) k^* \quad (9.51)$$

It will be larger than (3.18) not only because $k^*$ is larger, but also because the term deducting required reserves is now smaller.

If not, somewhere $dM_t = 0$ will be hit and the aggregate money stock fixed. Afterwards,

$$(H / M)^* = [1 - (1 – h) (1 – f)] / [1 – h (1 - \gamma)] \quad (9.52)$$

The economy will then follow the path (9.34) and (9.35) tending towards

$$f'(k^*) = [(1 + n) - \rho (1 – d)] / \{ \rho [1 – g' – rr' (H/M)^*] + \rho^2 [rr' (H/M)^* (1 – d_r) / (1 + n) + g' (1 – d_y) / (1 + n)] \} \quad (9.53)$$

$$c^* = f(k^*) \{1 - rr' (H/M)^* [1 – (1 – d_r) / (1 + n)] - g' [1 – (1 – d_y) / (1 + n)]\} – (n + d) k^* \quad (9.54)$$

We can examine again variants and special cases when the first-difference equation (9.9) with $a = 0$ holds.

1) Let $g' = 0$; then, $H_1$ and $M_1$ – in a policy where inventories are set to zero – can be obtained from:

$$0 = z_1 = z_0 (1 - d_h) / (1 + n) + g'' [M_1 - M_0 / (1 + n)] f(k_0) / M_1$$

$$M_1 = (1 - h)(1 - f) M_0 / (1 + n) + [1 - h (1 - \gamma)] H_1$$

For $t = 2,3,\ldots$ $z_t = 0$ imply:

$$g'' [M_t - M_{t-1} / (1 + n)] f(k_{t-1}) / M_t = 0$$

and therefore, determines $M_t = M_{t-1} / (1 + n)$; the monetary base multiplier sets then $H_t$ such that

$$M_t = (1 - h)(1 - f) M_{t-1} / (1 + n) + [1 - h (1 - \gamma)] H_t$$

i.e.,

$$H_t / M_t = [1 - (1 - h)(1 - f)] / [1 - h (1 - \gamma)]$$

Then an immediate jump to the steady-state of (H / M) is accomplished. The economy follows (9.29) and (9.30).

2) Let $g' = 0$. If a lagged term $h'' (1 - d_h) / (1 + n)\, dM_{t-1} / P_{t-1}$ were added to the capital state equation – deducted from the inventory one -, the zero inventory policy can be accomplished if the central authority targets $M_1$ such that

$$0 = z_1 = z_0 (1 - d_h) / (1 + n) + g'' [M_1 - M_0 / (1 + n)] f(k_0) / M_1 - h'' (1 - d_h) / (1 + n)\, dM_0 / P_0$$

fixing thus $H_1$ such that

$$M_1 = (1 - h)(1 - f) M_0 / (1 + n) + [1 - h (1 - \gamma)] H_1$$

For $t = 2,3\ldots$ $z_t = 0$ the authority sets $M_t$ such that:

$$g'' [M_t - M_{t-1} / (1 + n)] f(k_{t-1}) / M_t = h'' (1 - d_h) / (1 + n) [M_{t-1} - M_{t-2} / (1 + n)] f(k_{t-2}) / M_{t-1}$$

or

$[M_t - M_{t-1} / (1 + n)] / M_t = [f(k_0)/ f(k_{t-1})]\ [(h''/g'') (1 - d_h) / (1 + n)]^{t-1}\ \{[M_1 - M_0 / (1 + n)] / M_1\}$

or

$M_t = [M_{t-1} / (1 + n)] / (1 - [f(k_0)/ f(k_{t-1})]\ [(h''/g) (1 - d_h) / (1 + n)]^{t-1}\ \{[M_1 - M_0 / (1 + n)] / M_1\})$

systematically requiring $H_t$:

$H_t = [M_t - (1 - h) (1 - f) M_{t-1} / (1 + n)] / [1 - h (1 - \gamma)]$

3) Again, if $g' [f(k_{t-1}) - f(k_{t-2}) (1 - d_y) / (1 + n)]$ represents pure loss and do not affect inventories:

$f'(k^*) = [(1 + n) - \rho (1 - d)] / \{\rho [1 - g' - rr' (H/M)^*] + \rho^2 [rr' (H/M)^* (1 - d_r) / (1 + n) + g' (1 - d_y) / (1 + n)]\}$

With the first-order multiplier (9.12) – replacing (9.42) -, the dynamic properties of the optimal path would not change much. (9.29) and (9.30) still hold with a zero inventory policy; it can be accomplished if the central authority targets $M_1$ such that

$0 = z_1 = z_0 (1 - d_h) / (1 + n) + g'' [M_1 - M_0 / (1 + n)] f(k_0) / M_1 + g'[f(k_0) - f(k_{-1}) (1 - d_h) / (1 + n)]$

fixing thus $H_1$ such that

$M_1 = [1 - h (1 - \gamma)] (H_1 + \{[(1 - h) (1 - f)] / [1 - (1 - h) (1 - f)]\} H_0 / (1 + n))$

For t = 2,3... $z_t = 0$ the authority sets $M_t$ such that:

$g'' f(k_{t-1}) [M_t - M_{t-1} / (1 + n)] / M_t = - g' [f(k_{t-1}) - f(k_{t-2}) (1 - d_h) / (1 + n)]$ (9.55)

or

$M_t$ = [$M_{t-1}$ / (1 + n)] / (1 + (g'/g") {1 – [f($k_{t-2}$)/ f($k_{t-1}$)] (1 – $d_h$) / (1 + n)}) (9.56)

systematically requiring $H_t$ derived from (9.12):

$H_t$ = $M_t$ / [1 – h (1 - γ)]- {[(1 – h) (1 – f)] / [1 - (1 – h) (1 – f)]} $H_{t-1}$ / (1 + n)) (9.57)

Then it is possible to converge to a growth rate m* = 1 / [1 + n + (g'/g") (n + $d_h$)] – 1 < 1 / (1 + n) – 1, and to (9.49):

($H_t$ / $M_t$)* = (1 - (1 – h) (1 – f) {1 + (g'/g") [(n + $d_h$) / (1 + n)]}) / [1 – h (1 - γ)] (9.58)

provided it is larger than 0. In that case (9.50) still applies:

f'(k*) = [(1 + n) - ρ (1 – d)] / {ρ [1 – rr' (H/M)*] + $\rho^2$ [rr' (H/M)* (1 – $d_r$) / (1 + n) ]} (9.59)

Special cases would suggest the same deviations in the short-run monetary path relative to those of the other (first-order) multiplier, but not in the long run.

In any case, vanishing per capita nominal money balances – and also monetary base – are not avoided…

### 9.3.2. Taste for Real-Nominal Balance

If we include money – nominal per capita money balances – in the utility function, the problem supports easily a second-order differenced multiplier. The planner's problem becomes:

$$\underset{c_t,k_t,dH_t,H_t,dM_t,M_t,P_t}{Max} \sum_{t=1}^{\infty} \rho^t \; U(c_t, M_t)$$

s.t: (1 + n) $k_t$ = (1 – d) $k_{t-1}$ + f($k_{t-1}$) – $c_t$ – g" $dM_t/P_t$ – g' [f($k_{t-1}$) - f($k_{t-2}$) (1 – $d_y$) / (1 + n)] – rr' [$H_t/P_t$ – $H_{t-1}/P_{t-1}$ (1 - $d_r$) / (1 + n)] (9.55)

$dM_t$ = (1 – h) (1 – f) $dM_{t-1}$ / (1 + n) + [1 – h (1 - γ)] $dH_t$ (9.56)

$H_t$ = $H_{t-1}$ / (1 + n) + $dH_t$ (9.57)

$M_t$ = $M_{t-1}$ / (1 + n) + $dM_t$ (9.58)

$M_t$ = $P_t$ f($k_{t-1}$) (9.59)

$c_t \geq 0$, $k_t \geq 0$, $M_t \geq 0$, $H_t \geq 0$ (, $z_t \geq 0$), $- f(k_{t-2}) / (1 + n) \leq dM_t / P_t \leq f(k_{t-1})$

Given $k_{-1}$, $k_0$, $H_0$, $M_0$, $dM_0$, $P_0$(, $z_0$)

The inventory equation can now become redundant – (at least around the steady state) as long as m* > 1 / (n + 1) – 1; we therefore ignore it.

The Hamiltonian analog would be linear in $dH_t$. If $H_t$ is just dictated by $M_t$ and not a corner (i.e., different from 0), we must (may… we are applying rules of a first order Hamiltonian…) be in the presence of a singular solution for H. The problem can be further simplified to:

$$\max_{k_t, H_t, M_t} \sum_{t=1}^{\infty} \rho^t \, U\{(1 - d)\, k_{t-1} + f(k_{t-1}) - (1 + n)\, k_t - g''\, f(k_{t-1})\, [M_t - M_{t-1} / (1 + n)]/M_t - g'\, [f(k_{t-1}) - f(k_{t-2})\, (1 - d_y) / (1 + n)] - rr'\, [f(k_{t-1})\, H_t / M_t - f(k_{t-2})\, (H_{t-1}/ M_{t-1})\, (1 - d_r) / (1 + n)], M_t\} \quad (9.60)$$

s.t.: $$M_t = M_{t-1} / (1 + n) + [(1 - h)\, (1 - f)/(1 + n)]\, [M_{t-1} - M_{t-2} / (1 + n)] + [1 - h\, (1 - \gamma)]\, [H_t - H_{t-1} / (1 + n)] \quad (9.61)$$

$c_t \geq 0$, $H_t \geq 0$, $M_t \geq 0$, $H_t \geq 0$, $M_t/M_{t-1} \geq f(k_{t-1}) / [(1 + n)\, f(k_{t-1}) + f(k_{t-2})]$

Given $k_{-1}$, $k_0$, $H_0$, $M_{-1}$, $M_0$(, $z_0$)

with lagrangean form:

$$\max_{k_t, H_t, M_t, \mu_t} L = \sum_{t=1}^{\infty} \rho^t \, U\{(1 - d)\, k_{t-1} + f(k_{t-1}) - (1 + n)\, k_t - g''\, f(k_{t-1})\, [M_t - M_{t-1} / (1 + n)]/M_t - g'\, [f(k_{t-1}) - f(k_{t-2})\, (1 - d_y) / (1 + n)] - rr'\, [f(k_{t-1})\, H_t / M_t - f(k_{t-2})\, (H_{t-1}/ M_{t-1})\, (1 - d_r) / (1 + n)], M_t\} + \sum_{t=1}^{\infty} \mu_t\, (M_t - \{M_{t-1} / (1 + n) + [(1 - h)\, (1 - f) / (1 + n)]\, [M_{t-1} - M_{t-2} / (1 + n)] + [1 - h\, (1 - ?)]\, [H_t - H_{t-1} / (1 + n)]\}\ ) \quad (9.62)$$

$$\frac{\partial W}{\partial k_t} = \rho^t \left( -(1+n)\, U_c(c_t, M_t) + \rho\, U_c(c_{t+1}, M_{t+1}) \left[(1-d) + f'(k_t) \{1 - g' - g'' [M_{t+1} - M_t / (1+n)] / M_{t+1} - rr'\, H_{t+1} / M_{t+1} \}\right] + \rho^2\, U_c(c_{t+2}, M_{t+2})\, f'(k_t) \left[ rr' (1-d_r) / (1+n)\, H_{t+1} / M_{t+1} + g' (1-d_y) / (1+n)\right] \right) = 0 \qquad (9.63)$$

or

$$f'(k_t) = [(1+n)\, U_c(c_t, M_t) - \rho\, U_c(c_{t+1}, M_{t+1}) (1-d)] / \Big(\rho\, U_c(c_{t+1}, M_{t+1}) \{1 - rr'\, H_{t+1} / M_{t+1} - g' - g'' [(1+n)\, m_{t+1} + n] / [(1 + m_{t+1}) (1+n)]\} + {} + \rho^2\, U_c(c_{t+2}, M_{t+2}) [g' (1-d_y) / (1+n) + rr'\, H_{t+1} / M_{t+1} (1-d_r) / (1+n)] \Big)$$

$$\frac{\partial W}{\partial H_t} = \rho^t \{ -rr'\, U_c(c_t, M_t) (1 / M_t)\, f(k_{t-1}) + \rho\, U_c(c_{t+1}, M_{t+1})\, rr'\, f(k_{t-1}) (1 / M_t) (1 - d_r) / (1+n)\} - \mu_t [1 - h (1-\gamma)] + \mu_{t+1} \{[1 - h (1-\gamma)] / (1+n)\} = 0 \qquad (9.64)$$

$$\frac{\partial W}{\partial M_t} = \rho^t \{ [-g''\, M_{t-1} / (1+n) + rr'\, H_t]\, U_c(c_t, M_t) (1 / M_t^2)\, f(k_{t-1}) + \rho\, U_c(c_{t+1}, M_{t+1}) \{ [g'' / (1+n)]\, f(k_t) / M_{t+1} - rr''\, f(k_{t-1}) (H_t / M_t^2) (1-d_r) / (1+n) + U_M(c_t, M_t) \} + \mu_t - \mu_{t+1} [(1-h) (1-f)] / (1+n) + \mu_{t+2} [(1-h) (1-f)] / (1+n)^2 = 0 \qquad (9.65)$$

The dynamics of the system could be stated in terms of $k_t$ and $m_t = M_t / M_{t-1} - 1$ and $l_t = H_t / H_{t-1} - 1$ using the two FOC and the identity by which $c_t$ was replaced, the capital state equation.

Dynamic characteristics of the balanced path of the problem are analogous to those of section 5.1., but exhibit a more complex pattern. In the steady-state, we expect m* to tend to 0 - and therefore, - as $a / (H_t\, L_t)$ tends now to zero – according to (9.9):

$$(H / M)^* = [1 - (1-h) (1-f) / (1+n)] / [1 - h (1-\gamma)] \qquad (9.66)$$

and from (9.63)

$$f'(k^*) = [(1+n) - \rho (1-d)] / \{\rho [1 - g' - g'' n / (1+n) - rr' (H/M)^*] + \rho^2 [rr' (H/M)^* (1-d_r) / (1+n) + g' (1-d_y) / (1+n)]\} \quad (9.67)$$

As $(H/M)^* < 1$ – and comparing with (3.17) for $rr = rr' + g'$ -, k* is expected to be higher with a high-powered money supply for a required (commercial) reserve ratio $h < 1$ (and $f < 1$), only if g" and population growth is not too high…

Consumption could be obtained from:

$$c^* = f(k^*) \{1 - rr' (H/M)^* [1 - (1-d_r) / (1+n)] - g' [1 - (1-d_y) / (1+n)] - g'' n / (1+n)\} - (n+d) k^* \quad (9.68)$$

It will be larger than (3.18) (provided g', g" = 0, or not too large) not only because k* is larger, but also because the term deducting required reserves is now smaller.

## 9.4. Money-in-Utility

Suppose we now wish to extend the model to also encompass savings and time deposit formation, i.e., M2. Those are however only means to transfer capital management to – or rather, through… - commercial banks.

Then, we no longer have that output equals real money balances – which means that the price level determination equation (1.2) is abandoned. One could replace it by

$$M_t = P_t [f(k_{t-1}) + (1-d) k_{t-1}] \quad (9.69)$$

All the wealth in the economy would be "monetized". But now, the finance constraint term representing transactions delay could only loosely be linked to changes in M – in M2… Rather, it could be linked to change in value of production.

The system dynamics would become more complicated but generate the same type of predictions.

But now, one can provide a rationale for wealth holdings as money to enter the felicity function along with non-wealth ones. Say that $P_t c_t$ must be held as money. Then, $M_t - P_t c_t$ is the amount held as time deposits and – neglecting official reserves etc. - $P_t k_t - M_t + P_t c_t$ is wealth held as securities. If there is no preference for monetized wealth, of course it won't ever be… But as there are now services for capital management transfer, one can assume a felicity function $U(c_t, M_t/P_t - c_t, k_t - M_t/P_t + c_t)$, with $U_j(c, r, k) > 0$, $j = c, r, k$, implying, in general form, $U(c_t, M_t/P_t, k_t)$, with $U_j(c, r, k) > 0$, $c, r, k$ – guaranteeing preference of consumption relative to wealth, and, to some extent, money over capital (even if only $U_c(c, r, k) > 0$ should be required – and potentially $U_k(c, r, k) = 0$, but then $U_r(c, r, k) = U_r(c, r) > 0$).

We further allow for the possibility that nominal money balances enter the felicity function, which appears with a fourth argument, $U(c_t, M_t/P_t, k_t, M_t)$. Notice that, because real money balances also appear as argument, taste for balanced real-nominal growth is compatible with a negative fourth derivative of U(.) and a negative $\frac{\partial U}{\partial M_t} = U_m(c_t, M_t/P_t, k_t, M_t) / P_t + U_M(c_t, M_t/P_t, k_t, M_t)$: we are reproducing the same effects as including $P_t$ as a fourth argument with positive fourth derivative, $U_P(c_t, M_t/P_t, k_t, P_t) > 0$, but not sufficiently positive to off-set its effect through real money balances, i.e., maintaining $\frac{\partial U}{\partial P_t} = - U_m(c_t, M_t/P_t, k_t, P_t) / P_t^2 + U_P(c_t, M_t/P_t, k_t, P_t) < 0$. The problem becomes [49]:

$$\underset{c_t,k_t,dH_t,H_t,dM_t,M_t,P_t}{Max} \sum_{t=0}^{\infty} \rho^t \; U(c_t, M_t/P_t, k_t, M_t) \qquad (9.70)$$

s.t: $(1 + n)\, k_t = (1 - d)\, k_{t-1} + f(k_{t-1}) - c_t - g'' [M_t - M_{t-1} / (1 + n)]/P_t - g' [f(k_{t-1}) - f(k_{t-2}) (1 - d_y) / (1 + n)] - rr' [H_t/P_t - H_{t-1}/P_{t-1} (1 - d_r) / (1 + n)]$ (9.71)

$$dM_t = (1 - h)(1 - f)\, dM_{t-1} / (1 + n) + [1 - h (1 - \gamma)]\, dH_t \qquad (9.72)$$

$$H_t = H_{t-1} / (1 + n) + dH_t \qquad (9.73)$$

$$M_t = M_{t-1} / (1 + n) + dM_t \qquad (9.74)$$

[49] For Portugal, 1949-1995 – using information from Pinheiro *et al.,* (1997) -, the coefficients of the regression (without intercept) (9.72) using M2 – M1 plus savings and time deposits – as the money aggregate were, respectively, 0.943682 – approaching (1 – h) (1 – f) for an annual revolving period - and -0.046057 (the second term is insignificant). The long run ratio (using per capita aggregates) (M2 / H) for 1953-1995 was 3.75289.

$z_t = z_{t-1} (1 – d_h) / (1 + n) + g'' [M_t - M_{t-1} / (1 + n)]/P_t + g' [f(k_{t-1}) - f(k_{t-2}) (1 – d_y) / (1 + n)]$ (9.75)

$c_t \geq 0, k_t \geq 0, P_t \geq 0, H_t \geq 0, 0 \leq M_t \leq P_t [f(k_{t-1}) + (1 – d) k_{t-1}]$

Given $k_{-1}, k_0, H_0, M_0, dM_0, P_0, z_0$

$P_t$ is constrained to adjust – administratively set – in such a way that upper bound for the real value of $M_t$ is all existing wealth [50]. Money transaction costs, reflected in inventory rotation, are represented by the term $g'' [M_t - M_{t-1} / (1 + n)]/P_t$.

The problem can be further simplified to:

$$\underset{k_t, H_t, M_t, P_t}{Max} \sum_{t=1}^{\infty} \rho^t \; U\{(1 – d) k_{t-1} + f(k_{t-1}) - (1 + n) k_t – g'' [M_t - M_{t-1} / (1 + n)]/P_t - g' [f(k_{t-1}) - f(k_{t-2}) (1 – d_y) / (1 + n)] – rr' [H_t / P_t – (H_{t-1}/ P_{t-1}) (1 – d_r) / (1 + n)], M_t/P_t, k_t, M_t\} \quad (9.76)$$

s.t.: $M_t = M_{t-1} / (1 + n) + [(1 – h) (1 – f)/(1 + n)] [M_{t-1} - M_{t-2} / (1 + n)] + [1 – h (1 - \gamma)] [H_t - H_{t-1} / (1 + n)]$ (9.77)

$z_t = z_{t-1} (1 – d_h) / (1 + n) + g'' [M_t - M_{t-1} / (1 + n)]/P_t + g' [f(k_{t-1}) - f(k_{t-2}) (1 – d_y) / (1 + n)]$ (9.78)

$k_t \geq 0, P_t \geq 0, H_t \geq 0, 0 \leq M_t \leq P_t [f(k_{t-1}) + (1 – d) k_{t-1}]$

Given $k_{-1}, k_0, H_0, M_{-1}, M_0, P_0, z_0$

If $U_M(c_t, M_t/P_t, k_t, M_t) \neq 0$, the inventory equation can become redundant. Then, we can write the Lagrangean of the problem as:

$$\underset{k_t, H_t, M_t, P_t, \mu_t}{Max} \; L = \sum_{t=1}^{\infty} \rho^t \; U\{(1 – d) k_{t-1} + f(k_{t-1}) - (1 + n) k_t – g'' [M_t - M_{t-1} / (1 + n)]/P_t - g' [f(k_{t-1}) - f(k_{t-2}) (1 – d_y) / (1 + n)] – rr' [H_t / P_t – (H_{t-1}/ P_{t-1}) (1 – d_r) / (1 + n)], M_t/P_t, k_t, M_t\} + \sum_{t=1}^{\infty}$$

[50] Notice that there is no direct price determination equation – neither the usual (1.2), nor (9.69) unless we hit the bound. $P_t$ is targeted such that it guarantees an adequate balance between monetized and non-monetized real wealth as dictated by the individuals' tastes or shape of utility function.

$\mu_t$ ($M_t$ - {$M_{t-1}$ / (1 + n) + [(1 – h) (1 – f) / (1 + n)] [$M_{t-1}$ - $M_{t-2}$ / (1 + n)] + [1 – h (1 - γ)] [$H_t$ - $H_{t-1}$ / (1 + n)] } ) (9.79)

FOC are:

$\frac{\partial W}{\partial k_t}$ = $\rho^t$ ( - (1 + n) $U_c(c_t, M_t/P_t, k_t, M_t)$ + ρ $U_c(c_{t+1}, M_{t+1}/P_{t+1}, k_{t+1}, M_{t+1})$ [(1 – d)+ f'($k_t$) (1 – g')] + $\rho^2$ $U_c(c_{t+2}, M_{t+2}/P_{t+2}, k_{t+2}, M_{t+2})$ f'($k_t$) g' (1 – $d_y$) + $U_k(c_t, M_t/P_t, k_t, M_t)$ ) = 0 (9.80)

$\frac{\partial W}{\partial H_t}$ = $\rho^t$ { - rr' $U_c(c_t, M_t/P_t, k_t, M_t)$ (1 / $P_t$) + ρ $U_c(c_{t+1}, M_{t+1}/P_{t+1}, k_{t+1}, M_{t+1})$ rr' (1 / $P_t$) (1 - $d_r$) / (1 + n)} - $\mu_t$ [1 – h (1 - γ)] + $\mu_{t+1}$ {[1 – h (1 - γ)] / (1 + n)} = 0 (9.81)

$\frac{\partial W}{\partial M_t}$ = $\rho^t$ {- g'' (1 / $P_t$) $U_c(c_t, M_t/P_t, k_t, M_t)$ + ρ $U_c(c_{t+1}, M_{t+1}/P_{t+1}, k_{t+1}, M_{t+1})$ [g'' / (1 + n)] (1 / $P_{t+1}$) + $U_m(c_t, M_t/P_t, k_t, M_t)$ / $P_t$ + $U_M(c_t, M_t/P_t, k_t, M_t)$ + $\mu_t$ - $\mu_{t+1}$ [(1 – h) (1 – f)] / (1 + n) + $\mu_{t+2}$ [(1 – h) (1 – f)] / $(1 + n)^2$ } = 0 (9.82)

$\frac{\partial W}{\partial P_t}$ = $\rho^t$ ( {g'' [$M_t$ - $M_{t-1}$ / (1 + n)] + rr' $H_t$} $U_c(c_t, M_t/P_t, k_t, M_t)$ (1 / $P_t^2$) - ρ $U_c(c_{t+1}, M_{t+1}/P_{t+1}, k_{t+1}, M_{t+1})$ rr'' $H_t$ (1 / $P_t^2$) (1 – $d_r$) / (1 + n) - $U_m(c_t, M_t/P_t, k_t, M_t)$ ($M_t$ / $P_t^2$) ) = 0 (9.83)

From (9.81) and (9.82):

$\rho^{-t}$ $\mu_t$ = {g'' (1 / $P_t$) $U_c(c_t, M_t/P_t, k_t)$ - ρ $U_c(c_{t+1}, M_{t+1}/P_{t+1}, k_{t+1})$ [g'' / (1 + n)] (1 / $P_{t+1}$) + $U_m(c_t, M_t/P_t, k_t)$ / $P_t$ + $U_M(c_t, M_t/P_t, k_t, M_t)$ } + $\rho^{-t}$ [$\mu_{t+1}$ + $\mu_{t+2}$ / (1 + n)] [(1 – h) (1 – f) / (1 + n)] = {g'' (1 / $P_t$) $U_c(c_t, M_t/P_t, k_t)$ - ρ $U_c(c_{t+1}, M_{t+1}/P_{t+1}, k_{t+1})$ [g'' / (1 + n)] (1 / $P_{t+1}$) + $U_m(c_t, M_t/P_t, k_t)$ / $P_t$ + $U_M(c_t, M_t/P_t, k_t,$

$M_t) \}\ + \rho\ [(1 - h)(1 - f)] / (1 + n)\ \{ - rr' U_c(c_{t+1}, M_{t+1}/P_{t+1}, k_{t+1}) (1 / P_{t+1})\ + \rho\, U_c(c_{t+2}, M_{t+2}/P_{t+2}, k_{t+2})\, rr' (1 / P_{t+1}) (1 - d_r) / (1 + n)\} / [1 - h (1 - \gamma)]$ (9.84)

Then, replacing (9.84) in (9.81), for example:

$$\{ - rr' U_c(c_t, M_t/P_t, k_t, M_t) (1 / P_t)\ + \rho\, U_c(c_{t+1}, M_{t+1}/P_{t+1}, k_{t+1}, M_{t+1})\, rr' (1 / P_t) (1 - d_r) / (1 + n)\}$$

$$= \rho^{-t} \mu_t [1 - h (1 - \gamma)]\ - \rho^{-t} \mu_{t+1}\ \{[1 - h (1 - \gamma)] / (1 + n)\} =$$

$$= ( [1 - h (1 - \gamma)]\ \{g'' (1 / P_t) U_c(c_t, M_t/P_t, k_t, M_t)\ - \rho\, U_c(c_{t+1}, M_{t+1}/P_{t+1}, k_{t+1}, M_t)\ [g'' / (1 + n)] (1 / P_{t+1}) + U_m(c_t, M_t/P_t, k_t, M_t) / P_t + U_M(c_t, M_t/P_t, k_t, M_t)\}\ + \rho\ [(1 - h)(1 - f)] / (1 + n)\ \{ - rr'\ U_c(c_{t+1}, M_{t+1}/P_{t+1}, k_{t+1}, M_{t+1})\ (1\ /\ P_{t+1})\ + \rho\ U_c(c_{t+2}, M_{t+2}/P_{t+2}, k_{t+2}, M_{t+2})\, rr' (1 / P_{t+1}) (1 - d_r) / (1 + n)\}\ ) - \rho ( [1 - h\ (1 - \gamma)]\ \{g''\ (1\ /\ P_{t+1})\ U_c(c_{t+1}, M_{t+1}/P_{t+1}, k_{t+1}, M_{t+1})\ - \rho\ U_c(c_{t+2}, M_{t+2}/P_{t+2}, k_{t+2}, M_{t+2})\ [g''\ /\ (1 + n)]\ (1\ /\ P_{t+2}) + U_m(c_{t+1}, M_{t+1}/P_{t+1}, k_{t+1}, M_{t+1}) / P_{t+1} + U_M(c_{t+1}, M_{t+1}/P_{t+1}, k_{t+1}, M_{t+1})\}\ + \rho\ [(1 - h)(1 - f)] / (1 + n)\ \{ - rr' U_c(c_{t+2}, M_{t+2}/P_{t+2}, k_{t+2}, M_{t+2}) (1 / P_{t+2})\ + \rho\, U_c(c_{t+3}, M_{t+3}/P_{t+3}, k_{t+3}, M_{t+3})\, rr' (1 / P_{t+2}) (1 - d_r) / (1 + n)\}\ )\ / (1 + n) \quad (9.85)$$

The steady-state dynamics of the system allow for constant c*, $(M_t/P_t)^*$, $(P_t/P_{t-1})^*$, $(H_t/M_t)^*$, and k*. Some of their features can be inferred after equations (from (9.80) and (9.83)):

$$(1 + n) - \rho [(1 - d) + f'(k^*) (1 - g')] - \rho^2 f'(k^*) g' (1 - d_y) / (1 + n) = U_k(c^*, M_t/P_t, k^*, M^*) / U_c(c^*, M_t/P_t, k^*, M^*) \quad (9.86)$$

$$g'' [M_t - M_{t-1} / (1 + n)] / M_t\ + rr' [1 - \rho (1 - d_r) / (1 + n)] (H_t/M_t)^* = U_m(c_t, M_t/P_t, k_t, M^*) / U_c(c^*, M_t/P_t, k^*, M^*) \quad (9.87)$$

$\{ - rr' + \rho\, rr' (1 - d_r) / (1 + n)\} = ( [1 - h (1 - \gamma)]\ \{g'' - \rho\ [g'' / (1 + n)] (P_t / P_{t+1}) + [U_m(c_t, M_t/P_t, k_t, M_t) + P_t U_M(c_t, M_t/P_t, k_t, M_t)]$

$/ U_c(c_t, M_t/P_t, k_t, M_t) \} + \rho \; [(1 – h) (1 – f)] / (1 + n) \; \{ - rr' (P_t / P_{t+1}) + \rho \, rr' (P_t / P_{t+1}) (1 - d_r) / (1 + n)\} \; ) –$

$- \rho ( [1 – h (1 - \gamma)] \{g'' (P_t / P_{t+1}) - \rho \; [g'' / (1 + n)] (P_t / P_{t+2}) + [U_m(c_{t+1}, M_{t+1}/P_{t+1}, k_{t+1}, M_{t+1}) + P_{t+1} \, U_M(c_{t+1}, M_{t+1}/P_{t+1}, k_{t+1}, M_{t+1})] / U_c(c_t, M_t/P_t, k_t, M_t) (P_t / P_{t+1})\} + \rho \; [(1 – h) (1 – f)] / (1 + n) \; \{ - rr' (P_t / P_{t+2}) + \rho \, rr' (P_t / P_{t+2}) (1 - d_r) / (1 + n)\} \; ) \; / (1 + n) =$

$= [1 - \rho \, P_t / P_{t+1} / (1 + n)] \; ( [1 – h (1 - \gamma)] \; \{ g'' - \rho \; [g'' / (1 + n)] (P_t / P_{t+1}) + [U_m(c_t, M_t/P_t, k_t, M_t) + P_t \, U_M(c_t, M_t/P_t, k_t, M_t)] / U_c(c_t, M_t/P_t, k_t, M_t) \} +$

$+ \; \rho \; [(1 – h) (1 – f)] / (1 + n) (P_t / P_{t+1}) \; [ - rr' + \rho \, rr' (1 - d_r) / (1 + n)] \; )$ (9.88)

and $(H/M)^*$ of (9.11), with $1 + m^*$ replaced by $(P_t/P_{t-1})^*$. Given the utility function, M, k, c and M/P should be constant; the only $m^*$ satisfying such requirement would be zero, implying that in fact (9.86) holds along with:

$(H / M)^* \; = \; [1 - (1 – h) (1 – f) / (1 + n)] / [1 – h (1 - \gamma)]$ (9.89)

$g'' [1 – 1 /(1 + n)] + \; rr' [1 - \rho (1 – d_r) / (1 + n)] (H/M)^* \; = U_m(c^*, M^*/P^*, k^*, M^*) / U_c(c^*, M^*/P^*, k^*, M^*)$ (9.90)

$\{ - rr' + \rho \, rr' (1 - d_r) / (1 + n)\} \; \{1 - [1 - \rho / (1 + n)] \, \rho \, [(1 – h) (1 – f)] / (1 + n)\}$

$= [1 - \rho / (1 + n)] \;\; ( [1 – h (1 - \gamma)] \;\; \{ g'' - \rho \;\; [g'' / (1 + n)] + [U_m(c^*, M^*/P^*, k^*, M^*) + P^* \, U_M(c^*, M^*/P^*, k^*, M^*)] / U_c(c^*, M^*/P^*, k^*, M^*) \} )$ (9.91)

From the last expression, we conclude that $[U_m(c^*, M^*/P^*, k^*, M^*) + P^* \, U_M(c^*, M^*/P^*, k^*, M^*)]$ – and therefore $U_M(c^*, M^*/P^*, k^*, M^*)]$ - would have to be negative for an interior solution.

If $U_M(c_t, M_t/P_t, k_t, M_t) = 0$ and therefore we can write $U(c_t, M_t/P_t, k_t, M_t) = U(c_t, M_t/P_t, k_t)$, the problem has a similar dynamic pattern as that of section 9.3.1. A zero inventory target leads to convergence awkwardness, solvable with a first-order monetary base multiplier [51] instead of (9.72). If then also g' = 0, we return to $m^* = 1 / (1 + n) - 1$.

[51] For Portugal, 1948-1995 – using information from Pinheiro *et al.,* (1997) -, the intercept of the first-order regression was found insignificant (p-value of 17.4%, 18.0% when per capita aggregates were used); coefficients of the regression (without intercept, 1954-1995) (9.9) using M2 – M1 plus savings and time deposits – in per capita terms were, respectively, 1.05587 – which should be smaller than 1… - and 0.255662.

# 10. (Other) Taxes, Public(ly Provided) Goods, and (Other) Debt: a Final Discussion

With homogeneous, infinitely lived individuals, the previous model is able to accommodate a "real" public sector. That is, underlying the provision of consumption, another good may be valued by consumers, $g_t$, which may be financed through (real) taxes $T_t$, or debt issuances, $dD_t/P_t$. If all the periodic product must be monetized (and production of g uses the same technology):

$$c_t + g_t + i_t + dM_t/P_t = f(k_{t-1})$$

If money is going to be market determined and it is meaningful to optimize with respect to $M_t$, $g_t = T_t + dD_t/P_t$. Yet, the allocation of the right hand-side has no effect in the model (provided $g_t$ is not a public good): the government (the individual…) has also to spend real resources to provide for $g_t$…

Costs – real costs - of money issuance could also be assumed. Yet, if a function of *real* per capita money balances, say, of $G(M_t/P_t)$ per capita, was periodically deducted from output, it would not alter the systems' dynamic properties either. One can say CIA (as PIA or TTP) assumes that costs are a function of new issuances only – $G(dM_t/P_t) = dM_t/P_t$ or $dy_t$ –, which would neglect maintenance costs of the overall system.

# 11. Conclusion

We analyzed the effects of interpreting money creation channels – say, nominal transfers or direct central bank purchases and lending – as having differentiated speed. Money was rationalized as a complete transaction device, and the finance constraint re-interpreted – enlarged - to encompass the purchase or borrowing of cash-balances by individuals. Official reserves were introduced. Cash balances and money creation through credit – fuelled by high-powered money - were distinguished.

Optimal monetary policies minimize idle inventory build-up, this induced by discontinuities in the demand for money creation and delays in the conversion process, or due to vertical ones in the physical production activity itself. Efficient and equilibrium solutions were distinguished - basic inefficiency of competitive factor price formation was highlighted. Q- theories of both investment and – now also – employment and money balances were recovered.

Possible convenience of the introduction of taste-for-inflation at the felicity or production function level was noted. Also, that presence of nominal per capita money balances, as nominal consumption value – along with real consumption – as arguments of the felicity function were able to produce stable prices in the long-run optimum.

Obvious extensions are the introduction of uncertainty and forward expectations, staggered contracts – both suggesting unemployment generation –, and application of the conversion-delay principle at the international trade finance level.

# Appendix A.

The replication or partition of the flow equation (1.6) over $T_t$ units of time may not involve proportionality for all the terms. One would expect such proportionality for expenditure items – $c_t$ and $i_t$. But firstly, the average product per unit of time may itself depend on the production span, so that $y_t = f(k_{t-1}, T_t)$. Secondly, terms measuring changes in stocks such as g” $dM_t/P_t$ should not rise proportionally to $T_t$ – probably, will not change with it.

Finally, there may be additional costs to be added to the expenditure – not accounted in (1.6) – directly affected by the choice of $T_t$; part of these costs maybe autonomous – $T_t\, g(T_t)$ -, part proportional to the average product or real money balances - $h(T_t)\, M_t/P_t = h(T_t)\, T_t\, f(k_{t-1}, T_t)$ -, and yet another parcel proportional to the change in stock from beginning to end of period - $v(T_t)\, [M_t - M_{t-1} / (1 + n)]/P_t$ (g” and v(T) may, in fact, interact; that is, v(T) may multiply g” – or replace it all together, depending on the interpretation.). To account for losses changes due to $dy_t$ we would add to the left hand-side equation – and we should consider $z(T_t)\, [T_t\, f(k_{t-1}, T_t) - T_{t-1}\, f(k_{t-2}, T_{t-1}) / (1 + n)]$ (to some extent we are compounding to such effect by considering that $f(k_{t-1}, T_t)$ depends on $T_t$.) – z(.) has correspondence to g’ + rr’ of the main text.

Then (1.6) – and later equalities - is replaced by:

$$T_t\, (c_t + i_t) + [g'' + v(T_t)]\, [M_t - M_{t-1} / (1 + n\, T_{t-1})]/P_t + T_t\, g(T_t) + h(T_t)\, M_t/P_t + z(T_t)\, [T_t\, f(k_{t-1}, T_t) - T_{t-1}\, f(k_{t-2}, T_{t-1}) / (1 + n\, T_{t-1})] = T_t\, y_t = T_t\, f(k_{t-1}, T_t) \tag{A.1}$$

Differently from the usual Baumol’s (1952) [52] transactions argument generating inventory money demand, we do not stress the flow of the stock from one side of the economic system to the other; in here, it is assumed that the stock flows “to itself” – from income earners to producers and then back to the former again. Instead, by relying on Clower’s constraint, we focus on currency (circulation) and model the time interval at which an increment in the stock is required by the whole system from the issuing authority, which coincides with the interval between transactions (or payments) themselves – its inverse, with the concept of money

[52] See also Jovanovic (1982).

velocity. Yet, we are assuming that for the time unit used – which is exogenous and assumed known and tied to the adequate $U(c_t)$ -, we know how $f(k_{t-1}, T_t)$ relates to $T_t$ of those units... Also, in practice, $T_t$* would coincide with the minimal time interval required for a bank deposit of any sort to pay interest, or for interest payments on a bank loan to be due in the economy – and within which no interest compounding would be generated [53].

The revolving period may also affect the relevant utility in various ways. From (A.1), it is defined in such a way that it represents the appropriate – to be optimized by rational agents - interval between money stock changes. Then, consumers may consume smoothly between $\sum_{u=1}^{t-1} T_u$ and $\sum_{u=1}^{t-1} T_u$ + $T_t$ even if purchases are not: the per unit of time consumption $c_t$ is repeated – $T_t$ times; then, the representative agent maximizes: $\sum_{t=1}^{\infty} \rho^{\sum_{u=0}^{t-2} T_{u+1}} \sum_{j=1}^{T_t} \rho^j$ $U(c_t)$ = $\sum_{t=1}^{\infty} \rho^{\sum_{u=0}^{t-2} T_{u+1}} \rho \frac{1-\rho^{T_t}}{1-\rho}$ $U(c_t)$ [54] ; or $\sum_{t=1}^{\infty} \rho^{\sum_{u=1}^{t} T_u} \sum_{j=1}^{T_{t+1}} \rho^j \; U(\frac{T_t c_t}{T_{t+1}})$ = $\sum_{t=1}^{\infty} \rho^{\sum_{u=1}^{t} T_u} \rho \frac{1-\rho^{T_{t+1}}}{1-\rho} \; U(\frac{T_t c_t}{T_{t+1}})$ ; or $\sum_{t=1}^{\infty} \rho^{\sum_{u=1}^{t} T_u} \sum_{j=1}^{T_{t+1}} \rho^j \; U(c_{t,j})$ with $\sum_{j=1}^{T_{t+1}} c_{t,j}$ = $T_t$ $c_t$. Another alternative is that they optimize $\sum_{t=1}^{\infty} \prod_{u=1}^{t} \rho^{T_u} U(T_t c_t)$ : the period also conditions the consumption span at the utility level.

Obviously, the per unit of time felicity functional may be itself dependent on the time span – negatively – and $\sum_{t=1}^{\infty} \rho^{\sum_{u=0}^{t-2} T_{u+1}} \rho \frac{1-\rho^{T_t}}{1-\rho}$ $U(c_t, T_t)$ or $\sum_{t=1}^{\infty} \prod_{u=1}^{t} \rho^{T_u} U(T_t c_t, T_t)$ be more

[53] One can argue that such unit of time should be the one to which felicity is referred to – if between 0 and $T_t$ transactions were allowed (possibly, with minimal time interval of one unit of time in which t is measured). That would then suggest a "term structure" of interest rates…

[54] Ignoring the fact that $T_t$ does not have to be an integer… Yet, the approximation is valid for continuous time.

adequate, with the derivative relative to the second argument negative. This extra refinement would just produce a consistent effect and therefore we shall not include it.

Consider the third functional and let nominal money stock eventually enter the felicity function. The planner's problem becomes:

$$\underset{c_t, k_t, dM_t, M_t, P_t, T_t}{Max} \sum_{t=1}^{\infty} \prod_{u=1}^{t} \rho^{T_u} \, U(T_t\, c_t, M_t) = \sum_{t=1}^{\infty} \rho^{\sum_{u=1}^{t} T_u} \, U(T_t\, c_t, M_t) \qquad \text{(A.2)}$$

s.t: $(1 + n\, T_t)\, k_t = (1 - d\, T_t)\, k_{t-1} + T_t\, f(k_{t-1}, T_t) - T_t\, c_t - T_t\, g(T_t) - h(T_t)\, M_t/P_t - [g'' + v(T_t)]\, [M_t - M_{t-1} / (1 + n\, T_{t-1})]/P_t - z(T_t)\, [T_t\, f(k_{t-1}, T_t) - T_{t-1}\, f(k_{t-2}, T_{t-1}) / (1 + n\, T_{t-1})]$ (A.3)

$M_t = M_{t-1} / (1 + n\, T_{t-1}) + dM_t$ (A.4)

$M_t = P_t\, T_t\, f(k_{t-1}, T_t)$ (A.5)

$z_t = (1 - d\, T_t)\, z_{t-1}\, (1 + n\, T_t) + [g'' + v(T_t)]\, [M_t - M_{t-1} / (1 + n\, T_{t-1})]/P_t$ (A.6)

$c_t \geq 0,\ M_t \geq 0,\ T_t \geq 0,\ z_t \geq 0$

Given $k_{-1}, k_0, M_0, T_0, z_0$

(We could equivalently add the term $z(T_t)\, [T_t\, f(k_{t-1}, T_t) - T_{t-1}\, f(k_{t-2}, T_{t-1}) / (1 + n\, T_{t-1})]$ to the inventory equation and impose $z_t \geq z(T_t)\, [T_t\, f(k_{t-1}, T_t) - T_{t-1}\, f(k_{t-2}, T_{t-1}) / (1 + n\, T_{t-1})]$. Implicitly we are assuming that there is another, separate, inventory equation ruling $dy_t$ that does not become binding – or that implicit losses are just deducted, inflicted, from current production…)

As nominal money enters felicity, we can neglect the inventory equation. The problem can be further simplified to:

$$\underset{k_t, M_t, T_t}{Max} \sum_{t=1}^{\infty} \rho^{\sum_{u=1}^{t} T_u} \, U\{(1 - d\, T_t)\, k_{t-1} + [1 - h(T_t) - g'' - v(T_t) - z(T_t)]\, T_t\, f(k_{t-1}, T_t) - (1 + n\, T_t)\, k_t - T_t\, g(T_t) + T_t\, [g'' + v(T_t)]\, [M_{t-1} / (1 + n\, T_{t-1})]\, f(k_{t-1}, T_t) / M_t + z(T_t)\, T_{t-1}\, f(k_{t-2}, T_{t-1}) / (1 + n\, T_{t-1}),\ M_t\} \qquad \text{(A.7)}$$

$c_t \geq 0,\ M_t \geq 0,\ T_t \geq 0,\ (z_t \geq 0)\ M_t/M_{t-1} \geq f(k_{t-1}) / [(1 + n)\, f(k_{t-1}) + f(k_{t-2})]$

$$\text{Given } k_{-1},\ k_0,\ M_0,\ T_0(,\ z_0)$$

(The restriction on inventories is mimicked by $M_t/M_{t-1} \geq f(k_{t-1}) / [(1 + n) f(k_{t-1}) + f(k_{t-2})]$. It will play no role in the analysis though…)

F.O.C., along with the restriction, require, for t = 1, 2, 3,…:

$$\frac{\partial W}{\partial k_t} = \rho^{\sum_{u=1}^{t} T_u} \big( - (1 + n\,T_t)\, U_c(T_t\, c_t, M_t) + \rho^{T_{t+1}}\, U_c(T_{t+1}\, c_{t+1}, M_{t+1})\, [\, (1 - d\, T_{t+1}) + f_k(k_t, T_{t+1})\, T_{t+1}\, \{1 - h(T_{t+1}) - g'' - v(T_{t+1}) - z(T_{t+1}) + [g'' + v(T_{t+1})]\, [M_t / (1 + n\,T_t)] / M_{t+1}\}\, ] + \rho^{T_{t+1}+T_{t+2}}\, U_c(T_{t+2}\, c_{t+2}, M_{t+2})\, z(T_{t+2})\, T_{t+1}\, f_k(k_t, T_{t+1}) / (1 + n\,T_{t+1}) \big) = 0 \qquad \text{(A.8)}$$

$$\frac{\partial W}{\partial M_t} = \rho^{\sum_{u=1}^{t} T_u} \{ - [g'' + v(T_t)]\, U_c(T_t\, c_t, M_t)\, (M_{t-1} / M_t^2)\, f(k_{t-1}, T_t)\, T_t / (1 + n\,T_{t-1}) + \rho^{T_{t+1}}\, [g'' + v(T_{t+1})]\, U_c(T_{t+1}\, c_{t+1}, M_{t+1})\, (1 / M_{t+1})\, f(k_t, T_{t+1})\, T_{t+1} / (1 + n\,T_t) + U_M(T_t\, c_t, M_t)\} = 0 \qquad \text{(A.9)}$$

$$\frac{\partial W}{\partial T_t} = \rho^{\sum_{u=1}^{t} T_u}\, U_c(T_t\, c_t, M_t)\, \big( - d\, k_{t-1} + [1 - h(T_t) - g'' - v(T_t) - z(T_t)]\, f(k_{t-1}, T_t) - n\, k_t - g(T_t) - T_t\, g'(T_t) - T_t\, [h'(T_t) + z'(T_t)]\, f(k_{t-1}, T_t) - v'(T_t)\, T_t\, [M_t - M_{t-1} / (1 + n\,T_{t-1})]\, f(k_{t-1}, T_t) / M_t + z'(T_t)\, T_{t-1}\, f(k_{t-2}, T_{t-1}) / (1 + n\,T_{t-1}) + \{[1 - h(T_t) - g'' - v(T_t) - z(T_t)] + [g'' + v(T_t)]\, [M_{t-1} / (1 + n\,T_{t-1})] / M_t\}\, T_t\, f_T(k_{t-1}, T_t) \big) + \rho^{\sum_{u=1}^{t+1} T_u}\, U_c(T_{t+1}\, c_{t+1}, M_{t+1})\, \{- n\, T_{t+1}\, [g'' + v(T_{t+1})]\, [M_t / (1 + n\,T_t)^2]\, f(k_t, T_{t+1}) / M_{t+1} - n\, z(T_{t+1})\, T_t\, f(k_{t-1}, T_t) / (1 + n\,T_t)^2 + z(T_{t+1})\, f(k_{t-1}, T_t) / (1 + n\,T_t) + z(T_{t+1})\, T_t\, f_T(k_{t-1}, T_t) / (1 + n\,T_t)\} + \ln(\rho) \sum_{j=t}^{\infty} [\, \rho^{\sum_{u=1}^{j} T_u}\, U(T_j\, c_j)] = 0 \qquad \text{(A.10)}$$

The dynamics of the system can be stated in terms of $k_t$ and $m_t = M_t / M_{t-1} - 1$:

$$f_k(k_t, T_{t+1}) = [(1 + n\, T_t)\, U_c(T_t\, c_t, M_t) - \rho^{T_{t+1}}\, U_c(T_{t+1}\, c_{t+1}, M_{t+1})\, (1 - d\, T_{t+1})] / (\rho^{T_{t+1}}\, U_c(T_{t+1}\, c_{t+1}, M_{t+1})\, T_{t+1}\, \{[1 - h(T_{t+1}) - g'' - v(T_{t+1}) - z(T_{t+1})] + [g'' + v(T_{t+1})]\, [M_t / (1 + n\, T_t)] / M_{t+1}\} + \rho^{T_{t+1}+T_{t+2}}\, U_c(T_{t+2}\, c_{t+2}, M_{t+2})\, z(T_{t+2})\, T_{t+1} / (1 + n\, T_{t+1})) \qquad \text{(A.11)}$$

$$[(1 + m_{t+1}) / (1 + m_t)]\, (T_t / T_{t+1})\, [(1 + n\, T_t) / (1 + n\, T_{t-1})] = [f(k_t, T_{t+1}) / f(k_{t-1}, T_t)]\ \rho^{T_{t+1}}\, U_c(T_{t+1}\, c_{t+1}, M_{t+1})]\, [g'' + v(T_{t+1})] / \{U_c(T_t\, c_t, M_t)\, [g'' + v(T_t)]\} + U_M(T_t\, c_t, M_t)\, M_{t+1}\, (1 + n\, T_t) / \{f(k_{t-1}, T_t)\, T_{t+1}\, U_c(T_t\, c_t, M_t)\, [g'' + v(T_t)]\} \qquad \text{(A.12)}$$

$$\rho^{\sum_{u=1}^{t} T_u}\, U_c(T_t\, c_t, M_t)\, (-d\, k_{t-1} + [1 - h(T_t) - g'' - v(T_t) - z(T_t)]\, f(k_{t-1}, T_t) - n\, k_t - g(T_t) - T_t\, g'(T_t) - T_t\, [h'(T_t) + z'(T_t)]\, f(k_{t-1}, T_t) - v'(T_t)\, T_t\, [M_t - M_{t-1} / (1 + n\, T_{t-1})]\, f(k_{t-1}, T_t) / M_t + z'(T_t)\, T_{t-1}\, f(k_{t-2}, T_{t-1}) / (1 + n\, T_{t-1}) + \{[1 - h(T_t) - g'' - v(T_t) - z(T_t)] + [g'' + v(T_t)]\, [M_{t-1} / (1 + n\, T_{t-1})] / M_t\}\, T_t\, f_T(k_{t-1}, T_t)) + \rho^{\sum_{u=1}^{t+1} T_u}\, U_c(T_{t+1}\, c_{t+1}, M_{t+1})\, \{-n\, T_{t+1}\, [g'' + v(T_{t+1})]\, [M_t / (1 + n\, T_t)^2]\, f(k_t, T_{t+1}) / M_{t+1} - n\, z(T_{t+1})\, T_t\, f(k_{t-1}, T_t) / (1 + n\, T_t)^2 + z(T_{t+1})\, f(k_{t-1}, T_t) / (1 + n\, T_t) + z(T_{t+1})\, T_t\, f_T(k_{t-1}, T_t) / (1 + n\, T_t)\} = -\ln(\rho) \sum_{j=t}^{\infty} [\rho^{\sum_{u=1}^{j} T_u}\, U(T_j\, c_j, M_j)] \qquad \text{(A.13)}$$

Consider the steady-state in which $m^* = 1 / (1 + n\, T^*) - 1$; or that $U(T_t\, c_t, M_t) = U(T_t\, c_t)$ (and therefore (A.12) would not be active…), which would imply it. Developing the last equation:

$$\rho^{tT^*}\, U_c(T^*\, c^*, M^*)\, \{ - (d + n)\, k^* + [1 - h(T^*) - g'' - v(T^*) - z(T^*)]\, f(k^*, T^*) - g(T^*) - T^*\, g'(T^*) - T^*\, h'(T^*)\, f(k^*, T^*) + [1 -$$

h($T_t$) - z($T_t$)] T* $f_T$(k*, T*) } + $\rho^{(t+1)T^*}$ $U_c$(T* c*, M*) {- n T* [g" + v(T*)] [1 / (1 + n T*)] f(k*, T*) – n T* z(T*) f(k*, T*) / (1 + n T*)$^2$ + z(T*) f(k*, T*) / (1 + n T*) + z(T*) T* $f_T$(k*, T*) / (1 + n T*)} = - ln(ρ) $\sum_{j=t}^{\infty}$ [ $\rho^{jT^*}$ U(T* c*, M*)] = - ln(ρ) U(T* c*, M*)

$$\frac{\rho^{tT^*}}{1-\rho^{T^*}} \qquad (A.14)$$

or

{ - (d + n) k* + [1 - h(T*) - g" - v(T*) - z(T*)] f(k*, T*) - g(T*) – T* g'(T*) – T* h'(T*) f(k*, T*) + [1 - h($T_t$) - z($T_t$)] T* $f_T$(k*, T*)} + $\rho^{T^*}$ {- n T* [g" + v(T*)] [1 / (1 + n T*)] f(k*, T*) – n T* z(T*) f(k*, T*) / (1 + n T*)$^2$ + z(T*) f(k*, T*) / (1 + n T*) + z(T*) T* $f_T$(k*, T*) / (1 + n T*)} = - $\frac{\ln(\rho)}{1-\rho^{T^*}}$ U(T* c*, M*) / $U_c$(T* c*, M*) (A.15)

In the steady-state:

$f_k$(k*, T*) = [(1 + n T*) - $\rho^{T^*}$ (1 – d T*)] / (T* $\rho^{T^*}$ { [1 – h(T*) - z(T*)] + $\rho^{T^*}$ z(T*) / (1 + n T*)} ) (A.16)

As h(T*) > 0, k* will tend to be smaller than in the absence of such – due to the cash conversion delay – costs (per unit of – optimal - revolving period duration…. Yet, T* may press $f_k$($k_t$, T*) down if T* > 1...

Let us examine now the alternative specification:

$$\underset{k_t, M_t, T_t}{Max} \sum_{t=1}^{\infty} \rho^{\sum_{u=0}^{t-2} T_{u+1}} \sum_{j=1}^{T_t} \rho^j \ U(c_t, M_t) = \sum_{t=1}^{\infty} \rho^{\sum_{u=0}^{t-2} T_{u+1}} \rho \frac{1-\rho^{T_t}}{1-\rho} \ U\{(1 -$$

d $T_t$) $k_{t-1}$ / $T_t$ + [1 - h($T_t$) - g" - v($T_t$) - z($T_t$)] f($k_{t-1}$, $T_t$) - (1 + n $T_t$) $k_t$ / $T_t$ - g($T_t$) + [g" + v($T_t$)] [$M_{t-1}$ / (1 + n $T_{t-1}$)] f($k_{t-1}$, $T_t$) / $M_t$ + z($T_t$) ($T_{t-1}$ / $T_t$) f($k_{t-2}$, $T_{t-1}$) / (1 + n $T_{t-1}$), $M_t$} (A.17)

$c_t \geq 0$, $M_t \geq 0$, $T_t \geq 0$, ($z_t \geq 0$,) $M_t/M_{t-1} \geq f(k_{t-1}) / [(1 + n) f(k_{t-1}) + f(k_{t-2})]$

Given $k_{-1}$, $k_0$, $M_0$, $T_0$ (, $z_0$)

Now the period duration FOC generates:

$$\rho^{\sum_{u=0}^{t-2} T_{u+1}} \rho \frac{1-\rho^{T_t}}{1-\rho} U_c(c_t, M_t) ( - (1 / T_t^2) k_{t-1} + (1 / T_t^2) k_t - g'(T_t) - [h'(T_t) + z'(T_t)] f(k_{t-1}, T_t) - v'(T_t) [M_t - M_{t-1} / (1 + n T_{t-1})] f(k_{t-1}, T_t) / M_t + z'(T_t) (T_{t-1}/T_t) f(k_{t-2}, T_{t-1}) / (1 + n T_{t-1}) - z(T_t) (T_{t-1} / T_t^2) f(k_{t-2}, T_{t-1}) / (1 + n T_{t-1}) + \{[1 - h(T_t) - g'' - v(T_t) - z(T_t)] + [g'' + v(T_t)] [M_{t-1} / (1 + n T_{t-1})] / M_t\} f_T(k_{t-1}, T_t) ) ) + \rho^{\sum_{u=0}^{t-1} T_{u+1}} \rho \frac{1-\rho^{T_{t+1}}}{1-\rho} U_c(c_{t+1}, M_{t+1}) \{- n [g'' + v(T_{t+1})] [M_t / (1 + n T_t)^2] f(k_t, T_{t+1}) / M_{t+1} - n z(T_{t+1}) (T_t/ T_{t+1}) f(k_{t-1}, T_t) / (1 + n T_t)^2 + z(T_{t+1}) (1 / T_{t+1}) f(k_{t-1}, T_t) / (1 + n T_t) + z(T_{t+1}) (T_t / T_{t+1}) f_T(k_{t-1}, T_t) / (1 + n T_t)\} = \ln(\rho) \rho^{\sum_{u=0}^{t-2} T_{u+1}} \rho \frac{1-\rho^{T_t}}{1-\rho} U(c_t, M_t) - \ln(\rho) \sum_{j=t+1}^{\infty} \rho^{\sum_{u=0}^{j-2} T_{u+1}} \rho \frac{1-\rho^{T_j}}{1-\rho} U(c_j, M_j) \quad (A.18)$$

or

$$(1-\rho^{T_t}) U_c(c_t, M_t) ( - (1 / T_t^2) k_{t-1} + (1 / T_t^2) k_t - g'(T_t) - [h'(T_t) + z'(T_t)] f(k_{t-1}, T_t) - v'(T_t) [M_t - M_{t-1} / (1 + n T_{t-1})] f(k_{t-1}, T_t) / M_t + z'(T_t) (T_{t-1}/T_t) f(k_{t-2}, T_{t-1}) / (1 + n T_{t-1}) - z(T_t) (T_{t-1} / T_t^2) f(k_{t-2}, T_{t-1}) / (1 + n T_{t-1}) + \{[1 - h(T_t) - g'' - v(T_t) - z(T_t)] + [g'' + v(T_t)] [M_{t-1} / (1 + n T_{t-1})] / M_t\} f_T(k_{t-1}, T_t) ) ) + \rho^{T_t} (1-\rho^{T_{t+1}}) U_c(c_{t+1}, M_{t+1}) \{- n [g'' + v(T_{t+1})] [M_t / (1 + n T_t)^2] f(k_t, T_{t+1}) / M_{t+1} - n z(T_{t+1}) (T_t/ T_{t+1}) f(k_{t-1}, T_t) / (1 + n$$

$T_t)^2$ + $z(T_{t+1})$ $(1 / T_{t+1})$ $f(k_{t-1}, T_t) / (1 + n T_t)$ + $z(T_{t+1})$ $(T_t / T_{t+1})$ $f_T(k_{t-1}, T_t) / (1 + n T_t)\}$ = $\ln(\rho)(1-\rho^{T_t})U(c_t, M_t)$ - $\ln(\rho)$

$$\sum_{j=t+1}^{\infty} [\rho^{\sum_{u=t-1}^{j-2} T_{u+1}} (1-\rho^{T_j}) U(c_j, M_j)]$$

In the steady-state, if $(M_t / M_{t-1})^* = 1 + m^* = 1 / (1 + n T^*)$ – say, $U(T_t c_t, M_t) = U(T_t c_t)$:

$$(1-\rho^{T^*}) (U_c(c^*, M^*) \{- g'(T^*) - h'(T^*) f(k^*, T^*) - [z(T^*) / T^*] f(k^*, T^*) / (1 + n T^*) + [1 - h(T^*) - z(T^*)] f_T(k_{k^*}, T^*)\} ) + \rho^{T^*}(1-\rho^{T^*}) (- n \{[g'' + v(T^*)] + z(T^*) / (1 + n T^*)\} f(k^*, T^*) + [z(T^*) / T^*] f(k^*, T^*) + z(T^*) f_T(k^*, T^*) ) / (1 + n T^*) = \ln(\rho)$$

$$(1-\rho^{T^*}) - \ln(\rho) (1-\rho^{T^*}) \sum_{j=t+1}^{\infty} \rho^{\sum_{u=t-1}^{j-2} T^*} U(c^*, M^*) = \ln(\rho)$$

$$(1-\rho^{T^*}) U(c^*, M^*) - \ln(\rho)(1-\rho^{T^*}) U(c^*, M^*) \sum_{j=t+1}^{\infty} \rho^{T^*(j-t)} =$$

$$\ln(\rho) (1-\rho^{T^*}) U(c^*, M^*) - \ln(\rho)(1-\rho^{T^*}) U(c^*, M^*) \frac{\rho^{T^*}}{1-\rho^{T^*}}] \quad \text{(A.19)}$$

or

$$\{- g'(T^*) - h'(T^*) f(k^*, T^*) - [z(T^*) / T^*] f(k^*, T^*) / (1 + n T^*) + [1 - h(T^*) - z(T^*)] f_T(k_{k^*}, T^*)\} + \rho^{T^*} (- n \{[g'' + v(T^*)] + z(T^*) / (1 + n T^*)\} f(k^*, T^*) + [z(T^*) / T^*] f(k^*, T^*) + z(T^*) f_T(k^*, T^*) ) / (1 + n T^*) = \ln(\rho) [1 - \frac{\rho^{T^*}}{1-\rho^{T^*}}] U(c^*, M^*) / U_c(c^*, M^*) = \ln(\rho) \frac{1-2\rho^{T^*}}{1-\rho^{T^*}} U(c^*, M^*) / U_c(c^*, M^*) \quad \text{(A.20)}$$

One can use (A.20) to infer some properties of T*, the steady-state optimal "time-to-build". Suppose all costs were 0 and we are just left with the Clower's finance constraint term, g". Also, that average labor product is interval independent; then (A.20) becomes:

$$n\, g''\, f(k^*, T^*)\, U_c(c^*, M^*) / U(c^*, M^*) = \ln(\rho) \frac{2\rho^{T^*}-1}{\rho^{T^*}-\rho^{2T^*}} (1 + n\, T^*) \quad \text{(A.21)}$$

(A.21) can also be written as:

$$n\, g''\, [U_c(c^*, M^*)\, c^* / U(c^*, M^*)] = \ln(\rho) \frac{2\rho^{T^*}-1}{\rho^{T^*}-\rho^{2T^*}} (1 + n\, T^*) (1 - s^*) \quad \text{(A.22)}$$

where $s^* = 1 - c^* / f(k^*, T^*)$, the steady-state savings rate. $\frac{2\rho^{T^*}-1}{\rho^{T^*}-\rho^{2T^*}}$ decreases with T* provided $(\rho^{T^*}-\rho^{2T^*}) < 0.5$; then, at given n and g'', if the elasticity of the felicity function is constant and n small (but non-null) or negative, T* and s* move (with the shape of the production function…) in the same direction.

If n = 0 (or g'' = 0 – then the advantage in a particular T* comes only from the existence of the minimal or unitary discounted period implicit in the felicity function definition and an implicit time elapse till production becomes available for expenditure - during which interest rate compounding cannot be observed):

$$\rho^{T^*} = \frac{1}{2} \quad \text{or} \quad T^* = -\frac{\ln(2)}{\ln(\rho)} = 0.69315 \ln\left(\frac{1}{\rho}\right) \approx 0.69315\, (1/\rho - 1) \quad \text{(A.23)}$$

Notice that $(1/\rho - 1) = 1/[\rho'(1 + n)] - 1$ approximates the discount rate – or , the individual discount rate when future generations are valued equally minus the population growth rate (divided by 1 plus the latter).

Suppose population is stable and the appropriate time unit to measure $U(c_t, M_t)$ is the month (30 days); if (we observe that) the optimal revolving period is a year – 365 days – the monthly discount factor would be $0.5^{1/T^*} = 0.5^{30/365} = 0.944621461$. If the optimal revolving period is the month, and the appropriate unit to measure $U(c_t, M_t)$ is the day, the daily discount factor is $0.5^{1/30} =$

0.97716; if T* is the week, $0.5^{1/7} = 0.90572$. Conversely, for a given time unit t, T* decreases with ⍰ or increases with the discount rate, $(1 / \rho) - 1$.

From (A.23), the technical relation (A.5) in the steady-state becomes:

$$M_t = - P_t \frac{\ln(2)}{\ln(\rho)} f(k^*) \qquad \text{(A.24)}$$

Then the transactions demand for money increases with the discount rate.

Notice that the income-velocity of money for a time span D – i.e., such that $M V = P D f(k^*)$ - is $V = D / [\ln(2) \ln(1 / \rho)] = D / [\ln(2) \ln(1 / \rho_D^{1/D})]$ where $\rho_D$ is the discount factor applying to time span D. If one estimates income-velocity for a given D, one can infer $\rho_D$.

Also, if one estimates a money demand function, as $M_t = - P_t \frac{\ln(2)}{\ln(\rho)} [D' f(k^*)] / D' = [\ln(2) \ln(1 / \rho_{D'}^{1/D'})] P_t [D' f(k^*)] / D'$, one can find an interest elasticity of demand $dM_t / d(1/\rho_{D'}) (1/\rho_{D'} - 1) / M_t = (1/\rho_{D'} - 1) (1/D') (1 / \rho_{D'}^{(1/D'-2)}) / \ln(1 / \rho_{D'}^{1/D'})$.

# Appendix B

One can approximate an annual money creation equation from a mechanism revolving j (the inverse of money income velocity) aggregating over j consecutive periods and approximating the with the annual data

$$dM_t = [(1 – h) (1 – f)]^j dM_{t-1} / (1 + n) + \{[1 – (1 – h)^j] / h\} [1 – h (1 - \gamma)] [dH_t + dH_{t-1} / (1 + n)] / 2 \qquad (B.1)$$

Obviously, j becomes estimable.

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